Online Appendix

A Proofs

A.1 Proof of Lemma 3: Properties of the Efficiency of Distribution

We provide a proof of Lemma 3 in this section before proving Propositions 1 and 2 in the next two sections, even though the latter come earlier in the main text. We do this because Lemma 3 does not depend on those propositions, but the fact that $\kappa(\cdot)$ is increasing and concave is useful in the proofs of those propositions.

We first prove Lemma 3 for the case in which space is one dimensional, followed by the case in which space is two dimensional.

A.1.1 One Dimension

First, define $\tilde{t}(\delta) \equiv t(\delta)^{1-\varepsilon}$. The integral of the function $\tilde{t}\left(|s|\right)$ over a line segment of length x can be expressed as

$$g(x) = \int_{-x/2}^{x/2} \tilde{t}\left(|\delta|\right) d\delta = 2\int_{0}^{x/2} \tilde{t}(\delta) d\delta.$$

It will sometimes be convenient to change variables and express this as

$$g(x) = x\int_{0}^{1} \tilde{t}\left(\frac{x}{2}u\right) du.$$

We thus have two expressions for the efficiency of distribution:

$$\kappa(n) = ng\left(\frac{1}{n}\right) = 2n\int_{0}^{\frac{1}{2n}} \tilde{t}(\delta) d\delta \tag{7}$$

$$\kappa(n) = ng\left(\frac{1}{n}\right) = \int_{0}^{1} \tilde{t}\left(\frac{u}{2n}\right) du \tag{8}$$

It will be useful to have expressions for the first and second derivative. Differentiating each with respect to n yields two expressions for the first derivative:

$$\kappa'(n) = 2\int_{0}^{\frac{1}{2n}} \tilde{t}(\delta) d\delta - \frac{1}{n}\tilde{t}\left(\frac{1}{2n}\right) \tag{9}$$

$$\kappa'(n) = \int_{0}^{1} \left[-\frac{1}{2n^2}\tilde{t}'\left(\frac{u}{2n}\right)\right] du \tag{10}$$

An expression for the second derivative comes from differentiating (9)

$$\kappa''(n) = \frac{1}{2n^3}\tilde{t}'\left(\frac{1}{2n}\right) \tag{11}$$

Claim A.1 *$\kappa(n) \equiv ng\left(\frac{1}{n}\right)$ is strictly increasing and strictly concave, and satisfies the following properties:*

1. *$\kappa(0) = 0$;*

2. *$\lim_{n\to\infty} \kappa(n) = 1$;*

3. *$1 - \kappa(n)$ follows a power law with exponent 1 as $n \to \infty$, i.e., $\lim_{n\to\infty} n\,[1-\kappa(n)] = -\frac{1}{4}\tilde{t}'(0) > 0$.*

Proof. $\tilde{t}$ is strictly decreasing because t is strictly increasing and $\varepsilon > 1$. As a result, (10) implies that κ is strictly increasing and (11) implies that κ is strictly concave. $\kappa(0) = 0$ follows from (8) and the fact that $\lim_{y\to\infty} t(y) = \infty$ which implies that $\lim_{y\to\infty} \tilde{t}(y) = 0$. $\lim_{n\to\infty} \kappa(n) = 1$ follows from (8) and $\tilde{t}(0) = 1$.

Beginning with (8), taking the limit as $n \to \infty$, using $x = 1/n$, and applying L'Hopital's rule gives

$$\begin{aligned}
\lim_{n\to\infty} n\,(1-\kappa(n)) &= \lim_{n\to\infty} n\left(1 - \int_0^1 \tilde{t}\left(\frac{u}{2n}\right) du\right) \\
&= \lim_{x\to 0} \frac{1 - \int_0^1 \tilde{t}\left(\frac{ux}{2}\right) du}{x} \\
&= \lim_{x\to 0} \frac{0 - \int_0^1 \frac{u}{2}\tilde{t}'\left(\frac{ux}{2}\right) du}{1} \\
&= -\frac{1}{4}\tilde{t}'(0)
\end{aligned}$$

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Claim A.2 *If $\lim_{\delta\to\infty} \tilde{t}(\delta)\delta = 0$ then $\kappa'(0) = 2\int_0^\infty \tilde{t}(\delta)d\delta$*

Proof. Taking the limit of the (9) gives

$$\begin{aligned}
\lim_{n\to 0} \kappa'(n) &= \lim_{n\to 0} 2\int_0^{\frac{1}{2n}} \tilde{t}(\delta)d\delta - \frac{1}{n}\tilde{t}\left(\frac{1}{2n}\right) \\
&= 2\int_0^\infty \tilde{t}(\delta)d\delta - 2\lim_{\delta\to\infty} \delta\tilde{t}\,(\delta)
\end{aligned}$$

■

Claim A.3 *If $\lim_{\delta\to\infty} \tilde{t}(\delta)\delta^2 = 0$, then $\kappa''(0) = 0$*

Proof. The second derivative of κ at zero is defined as $\kappa''(0) = \lim_{n\to 0} \frac{\kappa'(n) - \kappa'(0)}{n}$. $\lim_{\delta\to\infty} \tilde{t}(\delta)\delta^2 = 0$

implies $\lim_{\delta\to\infty}\tilde{t}(\delta)\delta = 0$, so Claim A.2 gives $\kappa'(0) = 2\int_0^\infty \tilde{t}(\delta)d\delta$. Using this along (9) gives

$$\begin{aligned}
\kappa''(0) &= \lim_{n\to 0}\frac{\kappa'(n)-\kappa'(0)}{n} \\
&= \lim_{n\to 0}\frac{2\int_0^{\frac{1}{2n}}\tilde{t}(\delta)d\delta + \frac{1}{n}\tilde{t}\left(\frac{1}{2n}\right) - 2\int_0^\infty \tilde{t}(\delta)d\delta}{n} \\
&= \lim_{n\to 0}\frac{\frac{1}{n}\tilde{t}\left(\frac{1}{2n}\right)}{n} + \lim_{n\to 0}\frac{2\int_0^{\frac{1}{2n}}\tilde{t}(\delta)\,d\delta - 2\int_0^\infty \tilde{t}(\delta)\,d\delta}{n}
\end{aligned}$$

Using L'Hopital's rule for the second term and then changing variables to $\delta = \frac{1}{2n}$ gives

$$\begin{aligned}
\kappa''(0) &= \lim_{n\to 0}\frac{1}{n^2}\tilde{t}\left(\frac{1}{2n}\right) + \lim_{n\to 0}\left[-2\frac{1}{2n^2}\tilde{t}\left(\frac{1}{2n}\right)\right] \\
&= 4\lim_{\delta\to\infty}\delta^2\tilde{t}(\delta) + \lim_{\delta\to\infty} -4\delta^2\tilde{t}(\delta) \\
&= 0
\end{aligned}$$

■

A.1.2 Two Dimensions

First, define $\tilde{t}(\delta) \equiv t(\delta)^{1-\varepsilon}$. A hexagon with area x has sides of length $l = \psi\sqrt{x}$, where $\psi \equiv 2^{1/2}3^{-3/4}$. The integral of the function $\tilde{t}(\|s\|)$ over a hexagon with area x can be expressed as

$$g(x) = \int_0^{\psi x^{1/2}} \varpi\left(\frac{\delta}{\psi x^{1/2}}\right)\tilde{t}(\delta)2\pi\delta d\delta$$

where $\varpi(r)$ is the fraction of circle with radius r that intersects with a hexagon with side length 1. That is, if $\alpha \equiv \sqrt{3}/2$ is the radius of the largest circle that can be inscribed in a hexagon with side length 1, then $\varpi(r) = 1$ for $r \in [0,\alpha]$, $\varpi'(r) < 0$ for $r \in (\alpha, 1)$, and $\varpi(1) = 0$.[40] We first rewrite g in a form that is easier

[40] What is ϖ? To get at this, for a hexagon with side length 1, a circle with radius $\delta = \sqrt{1-(1/2)^2} = \frac{\sqrt{3}}{2}$ will be fully inscribed. Consider a circle with radius between $\delta \in \left(\frac{\sqrt{3}}{2}, 1\right)$. What fraction of the circle is inside the hexagon? Consider two line segments, each emanating from the center of the hexagon to the border of the hexagon. One of length $\frac{\sqrt{3}}{2}$ which is perpendicular to the side of the hexagon, and one of length δ. The angle θ between the two satisfies $\cos(\theta) = \frac{\sqrt{3}/2}{\delta}$. The fraction of the circle of length δ that is outside the hexagon is therefore $\frac{12\theta}{2\pi}$. Therefore $\varpi(\delta) = \begin{cases} 1 & 0 \le \delta \le \sqrt{3}/2 \\ 1 - \frac{6}{\pi}\cos^{-1}\left(\frac{\sqrt{3}/2}{\delta}\right) & \sqrt{3}/2 \le \delta \le 1 \end{cases}$.

to manipulate. First, define $\tilde{t}(\delta) \equiv t(\delta)^{1-\varepsilon}$. We then can change variables

$$\begin{aligned} g(x) &= \int_0^{\psi\sqrt{x}} \varpi\left(\frac{\delta}{\psi\sqrt{x}}\right)\tilde{t}(\delta)2\pi\delta d\delta \\ &= \psi^2 x\int_0^1 \varpi(u)\tilde{t}\left(\psi\sqrt{x}u\right)2\pi u du \end{aligned}$$

This implies that

$$\kappa(n) = ng\left(\frac{1}{n}\right) = n\int_0^{\psi n^{-1/2}} \varpi\left(\frac{\delta}{\psi n^{-1/2}}\right)\tilde{t}(\delta)2\pi\delta d\delta \tag{12}$$

$$\kappa(n) = ng\left(\frac{1}{n}\right) = \psi^2\int_0^1 \varpi(u)\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du \tag{13}$$

It will be useful to have expressions for the first and second derivative. Differentiating with respect to n yields

$$\kappa'(n) = \psi^2\int_0^1 \varpi(u)\tilde{t}'\left(\psi n^{-1/2}u\right)\left(-\psi\frac{1}{2}n^{-3/2}u\right)2\pi u du \tag{14}$$

To find the second derivative, we change variables once more to get

$$\kappa'(n) = \int_0^{\psi n^{-1/2}} \varpi\left(\frac{\delta}{\psi n^{-1/2}}\right)\left[-\tilde{t}'(\delta)\right]\pi\delta^2 d\delta$$

Differentiating once more, using $\varpi(1) = 0$, and changing variables yields

$$\begin{aligned} \kappa''(n) &= \int_0^{\psi n^{-1/2}} \varpi'\left(\frac{\delta}{\psi n^{-1/2}}\right)\frac{\delta}{\psi}\frac{1}{2}n^{-\frac{1}{2}}\left[-\tilde{t}'(\delta)\right]\pi\delta^2 d\delta \\ &= \psi^3 n^{-5/2}\frac{\pi}{2}\int_0^1 \varpi'(u)u^3\left[-\tilde{t}'\left(\psi n^{-1/2}u\right)\right]du \end{aligned}$$

Using the fact that $\varpi'(r) = 0$ for $r \in (0, \alpha)$ gives

$$\kappa''(n) = \psi^3 n^{-5/2}\frac{\pi}{2}\int_\alpha^1 \varpi'(u)u^3\left[-\tilde{t}'\left(\psi n^{-1/2}u\right)\right]du \tag{15}$$

Claim A.4 $\kappa(n) \equiv ng\left(\frac{1}{n}\right)$ *is strictly increasing and strictly concave, and satisfies the following properties:*

1. $\kappa(0) = 0$;
2. $\lim_{n\to\infty}\kappa(n) = 1$;
3. $1-\kappa(n)$ *follows a power law with exponent* $\frac{1}{2}$ *as* $n \to \infty$, *i.e.,* $\lim_{n\to\infty}\sqrt{n}\left[1-\kappa(N)\right] = -\tilde{t}'(0)\frac{\sqrt{2}}{3^{3/4}}\left(\frac{1}{3}+\frac{\ln 3}{4}\right) > 0$.

Proof. (14) implies that κ' is strictly positive because $t' > 0$ and $\varepsilon > 1$ imply that $\tilde{t}' < 0$. (15) implies that κ'' is strictly negative because ϖ' is strictly negative on $(\alpha, 1)$. $\kappa(0) = 0$ follows from (13) and the fact that $\lim_{y\to\infty} t(y) = \infty$ which implies that $\lim_{y\to\infty} \tilde{t}(y) = 0$. $\lim_{n\to\infty} \kappa(n) = 1$ follows from (13) and the facts that $\tilde{t}(0) = 1$, and $\psi^2 \int_0^1 \varpi(u)2\pi u du = 1$.

Beginning with (13), we can express $\sqrt{n}[1 - \kappa(n)]$ as $\sqrt{n}\left(1 - \psi^2 2\pi \int_0^1 \varpi(u)\tilde{t}\left(\frac{\psi u}{\sqrt{n}}\right) udu\right)$. Taking the limit as $n \to \infty$, using $x = \sqrt{n}$, and using L'Hopital's rule gives

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n}\,(1 - \kappa(n)) &= \lim_{n\to\infty} \sqrt{n}\left(1 - \psi^2 2\pi \int_0^1 \varpi(u)\tilde{t}\left(\frac{\psi u}{\sqrt{n}}\right) udu\right) \\
&= \lim_{x\to 0} \frac{1 - \psi^2 2\pi \int_0^1 \varpi(u)\tilde{t}\left(\psi u x\right) udu}{x} \\
&= \lim_{x\to 0} \frac{-\psi^2 2\pi \int_0^1 \varpi(u)\tilde{t}'\left(\psi u x\right) \psi u^2 du}{1} \\
&= \left[-\tilde{t}'(0)\right] \psi^2 2\pi \int_0^1 \varpi(u)\psi u^2 du
\end{aligned}$$

The result follows from the fact that $\psi^2 2\pi \int_0^1 \varpi(u)\psi u^2 du = \psi\left(\frac{1}{3} + \frac{\ln 3}{4}\right) = 2^{1/2}3^{-3/4}\left(\frac{1}{3} + \frac{\ln 3}{4}\right)$ ■

Before proceeding, it will be useful to derive an alternative expression for $\kappa'(n)$. Differentiating (12) with respect to n yields

$$\begin{aligned}
\kappa'(n) &= \int_0^{\psi n^{-1/2}} \varpi\left(\frac{\delta}{\psi n^{-1/2}}\right)\tilde{t}(\delta)2\pi\delta d\delta \\
&\quad + n\varpi(1)\tilde{t}\left(\psi n^{-1/2}\right)2\pi\psi n^{-1/2}\left(-\frac{1}{2}\right)\psi n^{-3/2} \\
&\quad + n\int_0^{\psi n^{-1/2}} \varpi'\left(\frac{\delta}{\psi n^{-1/2}}\right)\frac{\delta}{\psi}\frac{1}{2}n^{-1/2}\tilde{t}(\delta)2\pi\delta d\delta
\end{aligned}$$

Noting that $\varpi(1) = 0$ and changing variables gives

$$\begin{aligned}
\kappa'(n) &= \int_0^{\psi n^{-1/2}} \left[\varpi\left(\frac{\delta}{\psi n^{-1/2}}\right) + \frac{1}{2}\varpi'\left(\frac{\delta}{\psi n^{-1/2}}\right)\frac{\delta}{\psi n^{-1/2}}\right]\tilde{t}(\delta)2\pi\delta d\delta \\
&= \frac{\psi^2}{n}\int_0^1 \left[\varpi(u) + \frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du
\end{aligned}$$

We can separate this into two terms, the integral over $u \in [0, \alpha]$ and the integral from $[\alpha, 1]$. For $u \in [0, \alpha)$, $\varpi(u) = 1$ and $\varpi'(u) = 0$, so we can express the integral as

$$\kappa'(n) = \frac{\psi^2}{n}\int_0^\alpha \tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du + \frac{\psi^2}{n}\int_\alpha^1 \left[\varpi(u) + \frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du \tag{16}$$

Claim A.5 *If* $\lim_{\delta\to\infty} \tilde{t}(\delta)\delta^2 = 0$ *then* $\kappa'(0) = \int_0^\infty \tilde{t}(\delta)2\pi\delta d\delta$

Proof. Taking the limit of the first term of (16) gives

$$\lim_{n\to 0}\frac{\psi^2}{n}\int_0^{\alpha}\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du=\lim_{n\to 0}\int_0^{\alpha\psi n^{-1/2}}\tilde{t}(\delta)2\pi\delta d\delta=\int_0^{\infty}\tilde{t}(\delta)2\pi\delta d\delta$$

The second term of (16) can be expressed as

$$\frac{\psi^2}{n}\int_{\alpha}^{1}\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du=\int_{\alpha}^{1}\left[\frac{\varpi\left(u\right)}{u}+\frac{1}{2}\varpi'(u)\right]\tilde{t}\left(\psi n^{-1/2}u\right)\left(\psi n^{-1/2}u\right)^2 2\pi du$$

We next show that the limit of this second term is zero. If $\lim_{x\to\infty}\tilde{t}(x)x^2=0$, then $\tilde{t}(x)x^2$ has a peak. call it $\bar{r}$. Then the function $\left[\frac{\varpi(u)}{u}+\frac{1}{2}\varpi'\left(u\right)\right]\tilde{t}\left(\psi n^{-1/2}u\right)\left(\psi n^{-1/2}u\right)^2 2\pi$ is dominated by $\left[\frac{\varpi(u)}{u}+\frac{1}{2}\varpi'(u)\right]\bar{r}2\pi$. Since the latter is integrable on $[\alpha,1]$ ($\int_{\alpha}^{1}\left[\frac{\varpi(u)}{u}+\frac{1}{2}\varpi'(u)\right]du\leq\int_{\alpha}^{1}\left[\frac{1}{u}+\frac{1}{2}\varpi'(u)\right]du=\ln\frac{1}{\alpha}-\frac{1}{2}$) Dominated convergence means we can bring the limit inside the integral. Since $\lim_{n\to 0}\tilde{t}\left(\psi n^{-1/2}u\right)\left(\psi n^{-1/2}u\right)^2=0$, the limit of the second terms is zero. ■

Claim A.6 *If* $\lim_{\delta\to\infty}\tilde{t}(\delta)\delta^4=0$*, then* $\kappa''(0)=0$

Proof. The second derivative of κ at zero is defined as $\kappa''(0)=\lim_{n\to 0}\frac{\kappa'(n)-\kappa'(0)}{n}$. $\lim_{\delta\to\infty}\tilde{t}(\delta)\delta^4=0$ implies $\lim_{\delta\to\infty}\tilde{t}(\delta)\delta^2=0$, so Claim A.5 gives $\kappa'(0)=\int_0^{\infty}\tilde{t}(\delta)2\pi\delta d\delta$. Using this along (16) gives

$$\begin{aligned}\kappa''(0)&=\lim_{n\to 0}\frac{\frac{\psi^2}{n}\int_0^1\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du-\int_0^{\infty}\tilde{t}\left(\delta\right)2\pi\delta d\delta}{n}\\&=\lim_{n\to 0}\frac{\frac{\psi^2}{n}\int_{\alpha}^1\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du}{n}+\frac{\frac{\psi^2}{n}\int_0^{\alpha}\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du-\int_0^{\infty}\tilde{t}\left(\delta\right)2\pi\delta d\delta}{n}\end{aligned}$$

We next show that each of the two terms is equal to zero. We first rearrange the first term and take the limit inside the integral using dominated convergence (the function $\left[-\left(\varpi(u)+\frac{1}{2}\varpi'(u)u\right)\right]\frac{2\pi u}{u^4}$ is integrable on the domain $u\in[\alpha,1]$, in particular $\int_{\alpha}^{1}\left[-\left(\varpi(u)+\frac{1}{2}\varpi'(u)u\right)\right]2\pi u du=4\alpha$, and the fact that $\lim_{y\to\infty}\tilde{t}(y)y^4=0$ implies that $\tilde{t}(y)y^4$ has a uniform upper bound)

$$\begin{aligned}&\lim_{n\to 0}\frac{\frac{\psi^2}{n}\int_{\alpha}^1-\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du}{n}\\=\;&\lim_{n\to 0}\frac{1}{\psi^2}\int_{\alpha}^{1}-\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\tilde{t}\left(\psi n^{-1/2}u\right)\left(\psi n^{-1/2}u\right)^4\frac{1}{u^4}2\pi u du\\=\;&\frac{1}{\psi^2}\int_{\alpha}^{1}-\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\left[\lim_{n\to 0}\tilde{t}\left(\psi n^{-1/2}u\right)\left(\psi n^{-1/2}u\right)^4\right]\frac{2\pi u}{u^4}du\\=\;&\frac{1}{\psi^2}\int_{\alpha}^{1}-\left[\varpi(u)+\frac{1}{2}\varpi'(u)u\right]\left[\lim_{y\to\infty}\tilde{t}(y)y^4\right]\frac{2\pi u}{u^4}du\\=\;&\frac{4\alpha}{\psi^2}\lim_{y\to\infty}\tilde{t}\left(y\right)y^4\\=\;&0\end{aligned}$$

For the second term, we can change variables and use L'Hopital's rule.

$$\begin{aligned}
\lim_{n\to 0}\frac{\frac{\psi^2}{n}\int_0^{\alpha}\tilde{t}\left(\psi n^{-1/2}u\right)2\pi u du-\int_0^{\infty}\tilde{t}\left(\delta\right)2\pi\delta d\delta}{n} &= \lim_{n\to 0}\frac{\int_0^{\alpha\psi n^{-1/2}}\tilde{t}\left(\delta\right)2\pi\delta d\delta-\int_0^{\infty}\tilde{t}\left(\delta\right)2\pi\delta d\delta}{n}\\
&= \lim_{n\to 0}\tilde{t}\left(\alpha\psi n^{-1/2}\right)2\pi\alpha\psi n^{-1/2}\left(-\frac{1}{2}\alpha\psi n^{-3/2}\right)\\
&= -\frac{\pi}{(\alpha\psi)^2}\lim_{n\to 0}\tilde{t}\left(\alpha\psi n^{-1/2}\right)\left(\alpha\psi n^{-1/2}\right)^4\\
&= -\frac{\pi}{(\alpha\psi)^2}\lim_{y\to\infty}\tilde{t}(y)y^4\\
&= 0
\end{aligned}$$

Together, these imply that $\kappa''(0)=0$. ■

A.2 Proof of Proposition 1: Convergence of Profit Function

Define the function $G(x)$ to be the integral of $T\left(\|s\|\right)^{1-\varepsilon}$ over a line segment of length x if $\mathtt{d}=1$ or over a regular hexagon of area x if $\mathtt{d}=2$ centered at the origin. That is,

$$\begin{aligned}
G(x) &\equiv 2\int_0^{x/2}T(\delta)^{1-\varepsilon}d\delta, \quad \mathtt{d}=1\\
G(x) &\equiv \int_0^{\psi x^{1/2}}\varpi\left(\frac{\delta}{\psi x^{1/2}}\right)T(\delta)^{1-\varepsilon}2\pi\delta d\delta, \quad \mathtt{d}=2
\end{aligned}$$

where $\varpi(r)$ is the fraction of circle with radius r that intersects with a hexagon with side length 1 as in Appendix A.1.2. Notice that $G(x)=\Delta^{\mathtt{d}}g\left(\frac{x}{\Delta^{\mathtt{d}}}\right)$.[41] We begin by restating a well-known result from discrete geometry.

Theorem A.7 *(Theorem of L. Fejes Toth on sums of moments): Let $f:[0,\infty)\to\mathbb{R}$ be a nondecreasing function and let H be a convex 3,4,5, or 6-gon in $\mathbb{E}^2$. Then for any set of n points P in $\mathbb{E}^2$,*

$$\int_H \min\left\{f\left(\|x-p\|\right):p\in P\right\}dx\geq n\int_{H_n}f\left(\|x\|\right)dx$$

where H_n is a regular hexagon in $\mathbb{E}^2$ with area $|H|/n$ and center at the origin.

[41] This follows from the definition of t along with the change of variables $\tilde{\delta}=\frac{\delta}{\Delta}$. In one dimension, this gives:

$$G(x)=2\int_0^{x/2}t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon}d\delta=\Delta 2\int_0^{\frac{1}{\Delta}\frac{x}{2}}t\left(\tilde{\delta}\right)^{1-\varepsilon}d\tilde{\delta}=\Delta g\left(\frac{x}{\Delta}\right)$$

In two dimensions, this gives:

$$G(x)=\int_0^{\psi\sqrt{x}}\varpi\left(\frac{\delta}{\psi\sqrt{x}}\right)t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon}2\pi\delta d\delta=\Delta^2\int_0^{\frac{1}{\Delta}\psi\sqrt{x}}\varpi\left(\frac{\tilde{\delta}}{\frac{1}{\Delta}\psi\sqrt{x}}\right)t\left(\tilde{\delta}\right)^{1-\varepsilon}2\pi\tilde{\delta}d\tilde{\delta}=\Delta^2 g\left(\frac{x}{\Delta^2}\right)$$

The analogous statement in one dimension is straightforward.

Theorem A.8 *Let $f:[0,\infty)\to\mathbb{R}$ be a nondecreasing function and let L be a line segment in $\mathbb{E}$. Then for any set of n points P in $\mathbb{E}$,*

$$\int_L \min\{f(\|x-p\|):p\in P\}\,dx \geq n\int_{L_n} f(\|x\|)\,dx$$

where L_n is a line segment in $\mathbb{E}$ with length $|L|/n$ and center at the origin.

Proof. For the n points $\{p_k\}_{k=1}^n$, let ℓ_k be the set of points in the line segment L for which p_k is the closest. Each ℓ_k is a line segment. Let $\bar{x}_k$ and $\underline{x}_k$ denote the upper and lower endpoints of the line segment, and let $x_k^*=\frac{\bar{x}_k+\underline{x}_k}{2}$ denote its center.

Consider the line segment ℓ_k:

$$\begin{aligned}\int_{\underline{x}_k}^{\bar{x}_k} f(|x-p_k|) &= \int_0^{\bar{x}_k-p_k} f(u)du+\int_0^{p_k-\underline{x}_k} f(u)du\\ &=\int_0^{|\ell_k|/2} f(u)du+\int_{|\ell_k|/2}^{\bar{x}_k-p_k} f(u)du+\int_0^{|\ell_k|/2} f(u)du+\int_{|\ell_k|/2}^{p_k-\underline{x}_k} f(u)du\\ &=G(|\ell_k|)+\int_{|\ell_k|/2}^{\bar{x}_k-p_k} f(u)du+\int_{|\ell_k|/2}^{p_k-\underline{x}_k} f(u)du \end{aligned} \tag{17}$$

If $x_k^*\geq p_k$, which implies $\bar{x}_k-p_k\geq|\ell_k|/2\geq x_k^*-p_k$, the fact that f is increasing implies

$$\int_{|\ell_k|/2}^{\bar{x}_k-p_k} f(u)du \geq \int_{|\ell_k|/2}^{\bar{x}_k-p_k} f(u-(x_k^*-p_k))\,du=\int_{p_k-\underline{x}_k}^{|\ell_k|/2} f(v)dv$$

If, on the other hand, $x_k^*\leq p_k$, which implies $p_k-\underline{x}_k\geq|\ell_k|/2\geq p_k-x_k^*$, we have

$$\int_{|\ell|/2}^{p_k-\underline{x}_k} f(u)du \geq \int_{|\ell_k|/2}^{p_k-\underline{x}_k} f(u-(p_k-x_k^*))\,du=\int_{\bar{x}_k-p_k}^{|\ell_k|/2} f(v)dv$$

In either case, the sum of the final two terms of (17) are non-negative, giving

$$\int_{\underline{x}_k}^{\bar{x}_k} f(|x-p_k|)\geq G(|\ell_k|)$$

Note further that $G(x)=2\int_0^{x/2} f(u)du$ is convex because f non-decreasing implies that $G'(x)=f\left(\frac{x}{2}\right)$ is non-decreasing. Therefore, Jensen's inequality implies

$$\sum_{k=1}^n G(|\ell_k|)=n\left(\frac{1}{n}\sum_{k=1}^n G(|\ell_k|)\right)\geq nG\left(\frac{1}{n}\sum_{k=1}^n|\ell_k|\right)=nG\left(\frac{|L_n|}{n}\right)$$

■

We next apply these theorems to our context.

Lemma A.9 *For any k and any finite set of points $O_i \subset \mathcal{S}_i^k$,*

$$\int_{s\in\mathcal{S}_i^k} \max_{o\in O_i} T\left(\delta_{so}\right)^{1-\varepsilon} ds \leq |O_i| G\left(\frac{k^{\mathtt{d}}}{|O_i|}\right)$$

Proof. Since $T(\delta)$ is strictly increasing in δ, $T(\delta)^{1-\varepsilon}$ is strictly decreasing. The theorem of L. Fejes Toth on sums of moments and Theorem A.8 therefore imply that

$$\int_{s\in\mathcal{S}_i^k} \max_{o\in O_i} T\left(\delta_{so}\right)^{1-\varepsilon} ds = -\int_{s\in\mathcal{S}_i^k} \min_{o\in O_i}\left(-T\left(\delta_{so}\right)^{1-\varepsilon}\right) ds \leq -\left|O_i\right|\left(-G\left(\frac{k^{\mathtt{d}}}{|O_i|}\right)\right) = \left|O_i\right| G\left(\frac{k^{\mathtt{d}}}{|O_i|}\right)$$

■

The next result will be useful in deriving a lower bound for the firm's profit, by studying the profit delivered by a feasible but sub-optimal policy.

Lemma A.10 *For any $k > 0, N \in \mathbb{N}_0$*

$$\sup_{O_i\subseteq\mathcal{S}_i^k \,||O_i|=N} \int_{s\in\mathcal{S}_i^k} \max_{o\in O_i} T\left(\delta_{so}\right)^{1-\varepsilon} ds \geq NG\left(\rho(N)\frac{k^{\mathtt{d}}}{N}\right)$$

where $\rho(N) = \left(1 + \frac{3^{3/4}}{\sqrt{2N}}\right)^{-2}$.

Proof. We first consider the one-dimensional case. It is feasible to place the N points so that the line segment $\mathcal{S}_i^k$ with length k is divided into N segments each of length k/N with an element of O_i at the center of each line segment. Such a choice of O_i would deliver the value $NG\left(\frac{k}{N}\right)$. Since this value is weakly lower than the optimum and G is increasing, we have

$$\sup_{O_i\subseteq\mathcal{S}_i^k \,||O_i|=N} \int_{s\in\mathcal{S}_i^k} \max_{o\in O_i} T\left(\delta_{so}\right)^{1-\varepsilon} ds \geq NG\left(\frac{k}{N}\right) \geq NG\left(\rho(N)\frac{k}{N}\right)$$

We next turn two the case of two dimensions. As in the proof of the lemma above, define $\psi \equiv 2^{1/2}3^{-3/4}$. The set $\mathcal{S}_i^k$ is a square with side length k. It is sufficient to show that for any non-negative integer N, one can fit N regular hexagons with area $\left(1 + \frac{1}{\psi\sqrt{N}}\right)^{-2}\frac{k^2}{N}$ inside the square $\mathcal{S}_i^k$ as this would constitute a particular O_i choice. Since the side length of a hexagon with area x has side length of $\psi\sqrt{x}$, each of these hexagons would have a side length $l = \frac{\psi}{\sqrt{N}}\frac{1}{1+\psi^{-1}N^{-1/2}}k$. Since regular hexagons can form a regular tiling of the plane, we can consider hexagons each with side length l and tiling with $c = \left\lceil \psi\sqrt{3}\sqrt{N}\right\rceil$ columns and $r = \left\lceil \frac{1}{\psi\sqrt{3}}\sqrt{N}\right\rceil$ rows, where $\lceil x \rceil$ denotes the smallest integer weakly larger than x. Our proposed lattice has total width of $\left(\frac{3}{2}c+\frac{1}{2}\right)l$ and total height weakly less than $(2r+1)\sqrt{l^2-(l/2)^2}=\left(\sqrt{3}r+\frac{\sqrt{3}}{2}\right)l$ (with equality if there is more than one column). We first show that that total width is smaller than k, i.e., $\left(\frac{3}{2}c+\frac{1}{2}\right)l\le k$. To see this, we have

$$\begin{aligned}\left(\frac{3}{2}c+\frac{1}{2}\right)l &= \left(\frac{3}{2}\left\lceil\psi\sqrt{3}\sqrt{N}\right\rceil+\frac{1}{2}\right)\frac{\psi}{\sqrt{N}}\frac{1}{1+\psi^{-1}N^{-1/2}}k \\ &\le \left(\frac{3}{2}\left(\psi\sqrt{3}\sqrt{N}+1\right)+\frac{1}{2}\right)\frac{\psi}{\sqrt{N}}\frac{1}{1+\psi^{-1}N^{-1/2}}k \\ &= \frac{1+\frac{2\psi}{\sqrt{N}}}{1+\psi^{-1}N^{-1/2}}k \\ &\le k\end{aligned}$$

where the last step follows because $4\le 3^{3/2}$ implies $2\psi=\frac{2}{2^{-1/2}3^{3/4}}\le 3^{3/4}2^{-1/2}=\frac{1}{\psi}$.

We next show that the total height is less than k, i.e., $\left(\sqrt{3}r+\frac{\sqrt{3}}{2}\right)l\le k$. To see this, we have

$$\begin{aligned}\left(\sqrt{3}r+\frac{\sqrt{3}}{2}\right)l &= \left(\sqrt{3}\left\lceil\frac{1}{\psi\sqrt{3}}\sqrt{N}\right\rceil+\frac{\sqrt{3}}{2}\right)\frac{\psi}{\sqrt{N}}\frac{1}{1+\psi^{-1}N^{-1/2}}k \\ &\le \left(\sqrt{3}\left(\frac{1}{\psi\sqrt{3}}\sqrt{N}+1\right)+\frac{\sqrt{3}}{2}\right)\frac{\psi}{\sqrt{N}}\frac{1}{1+\psi^{-1}N^{-1/2}}k \\ &= \frac{1+\frac{1}{\psi\sqrt{N}}}{1+\psi^{-1}N^{-1/2}}k=k\end{aligned}$$

Finally, we note that such a lattice contains $cr=\left\lceil 2^{1/2}3^{-1/4}\sqrt{N}\right\rceil\left\lceil 2^{-1/2}3^{1/4}\sqrt{N}\right\rceil\ge N$ regular hexagons. It follows that N regular hexagons each with area $\left(1+\frac{1}{\psi\sqrt{N}}\right)^{-2}\frac{k^2}{N}$ fit inside the square $\mathcal{S}_i^k$. ■

Claim A.11 *For any k, Δ, $\bar{\pi}_j^{k\Delta}\ge\pi_j^{\Delta}$ where*

$$\bar{\pi}_j^{k\Delta}\equiv\sup_{\{N_i\ge 0\}}\sum_{i\in I^k}-N_i\underline{R}_i^k\xi+Z\left(q_j,\sum_{i'\in I^k}N_{i'}\right)^{\varepsilon-1}\sum_{i\in I^k}\bar{D}_i^k\bar{b}_i^kN_iG\left(\frac{k^{\mathtt{d}}}{N_i}\right)+Z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T(\delta)^{1-\varepsilon}d\delta$$

Proof. For any set of plants O, let O_i^k be the subset that are in square i. We begin with:

$$\begin{aligned}
\int_s D_s \max_{o\in O}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds &= \sum_{i\in I^k}\int_{s\in\mathcal{S}_i^k} D_s \max_{o\in O}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds \\
&\le \sum_{i\in I^k}\int_{s\in\mathcal{S}_i^k} D_s \sum_{i'\in I^k}\max_{o\in O_{i'}^k}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds \\
&\le \sum_{i\in I^k}\int_{s\in\mathcal{S}_i^k} \bar{D}_i^k \sum_{i'\in I^k}\bar{b}_{i'}^k \max_{o\in O_{i'}^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds \\
&= \sum_{i\in I^k}\bar{D}_i^k\bar{b}_i^k\int_{s\in\mathcal{S}_i^k}\max_{o\in O_i^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds + \sum_{i\in I^k}\sum_{i'\neq i}\bar{D}_i^k\bar{b}_{i'}^k\int_{s\in\mathcal{S}_i^k}\max_{o\in O_{i'}^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds
\end{aligned}$$

We can bound the first term using Lemma A.9. To bound the second term, note that

$$\int_{s\in\mathcal{S}_i^k}\max_{o\in O_{i'}^k} T(\delta_{so})^{1-\varepsilon} ds \le \int_{s\in\mathcal{S}_i^k}\max_{o\in\mathcal{S}_{i'}^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds.$$

The term $\int_{s\in\mathcal{S}_i^k}\max_{o\in O_{i'}^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds$ is maximized if $\mathcal{S}_i^k$, $\mathcal{S}_{i'}^k$ are contiguous, in which case,

$$\int_{s\in\mathcal{S}_i^k}\max_{o\in\mathcal{S}_{i'}^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds = k^{\mathtt{d}-1}\int_0^k T(\delta)^{1-\varepsilon} d\delta$$

In addition, $\sum_i\sum_{i'\neq i}\bar{D}_i^k\bar{b}_{i'}^k \le \frac{1}{k^2}\bar{D}\bar{b}$, so that $\sum_i\sum_{i'\neq i}\bar{D}_i^k\bar{b}_{i'}^k\int_{s\in\mathcal{S}_i^k}\max_{o\in O_{i'}^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds \le \bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T(\delta)^{1-\varepsilon}d\delta$. Together, these imply

$$\int_s D_s \max_{o\in O}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds \le \sum_i \bar{D}_i^k\bar{b}_i^k\int_{s\in\mathcal{S}_i^k}\left|O_i^k\right| G\left(\frac{k^{\mathtt{d}}}{\left|O_i^k\right|}\right) + \bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T(\delta)^{1-\varepsilon}d\delta$$

Similarly,

$$\sum_{o\in O} -R_o\xi = \sum_i\sum_{o\in O_i^k} -R_o\xi \le \sum_i\sum_{o\in O_i^k} -\underline{R}_i^k\xi = \sum_i -\left|O_i^k\right|\underline{R}_i^k\xi$$

Together, these imply that

$$
\begin{aligned}
\pi_j^{\Delta} &= \sup_O \sum_{o\in O} -R_o\xi + Z\left(q_j,|O|\right)^{\varepsilon-1}\int_s D_s \max_{o\in O}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds \\
&\leq \sup_O \sum_{i\in I^k} -\left|O_i^k\right| \underline{R}_i^k \xi + Z\left(q_j,|O|\right)^{\varepsilon-1}\left(\sum_{i\in I^k} \bar{D}_i^k \bar{b}_i^k \left|O_i^k\right| G\left(\frac{k^{\mathtt{d}}}{\left|O_i^k\right|}\right) + \bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T(\delta)^{1-\varepsilon} d\delta\right) \\
&\leq \sup_{\{N_i\in\mathbb{N}_0\}} \sum_{i\in I^k} -N_i \underline{R}_i^k \xi + Z\left(q_j, \sum_{i'\in I^k} N_{i'}\right)^{\varepsilon-1} \sum_{i\in I^k} \bar{D}_i^k \bar{b}_i^k N_i G\left(\frac{k^{\mathtt{d}}}{N_i}\right) + Z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T(\delta)^{1-\varepsilon} d\delta \\
&\leq \sup_{\{N_i\geq 0\}} \sum_{i\in I^k} -N_i \underline{R}_i^k \xi + Z\left(q_j, \sum_{i'\in I^k} N_{i'}\right)^{\varepsilon-1} \sum_{i\in I^k} \bar{D}_i^k \bar{b}_i^k N_i G\left(\frac{k^{\mathtt{d}}}{N_i}\right) + Z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T\left(\delta\right)^{1-\varepsilon} d\delta \\
&= \bar{\pi}_j^{k\Delta}
\end{aligned}
$$

where we use that Z is decreasing in N and slightly abuse notation so that N_i for a particular firm is choice of number of plants in $\mathcal{S}_i^k$. ■

Claim A.12 *Fix k. In the limit as $\Delta \to 0$,*

$$
\lim_{\Delta\to 0} \bar{\pi}_j^{k\Delta} \leq \bar{\pi}_j^k \equiv \sup_{n\geq 0} \int \left\{-n_s \underline{R}_s^k + z\left(q_j, \int n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1} \bar{D}_s^k \bar{b}_s^k \kappa\left(n_s\right)\right\} ds
$$

Proof. We begin with the definition of $\bar{\pi}_j^{k\Delta}$, use the expressions for G, T, Z, and ξ in terms of g, t, z, and Δ, and then substitute $n_i \equiv \frac{\Delta^{\mathtt{d}} N_i}{k^{\mathtt{d}}}$ to get

$$
\begin{aligned}
\bar{\pi}_j^{k\Delta} &\equiv \sup_{\{N_i\geq 0\}} \sum_{i\in I^k} \left\{-N_i \underline{R}_i^k \xi + Z\left(q_j,N\right)^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k N_i G\left(\frac{k^{\mathtt{d}}}{N_i}\right)\right\} + Z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k T(\delta)^{1-\varepsilon} d\delta \\
&= \sup_{\{N_i\geq 0\}} \sum_{i\in I^k} \left\{-N_i \underline{R}_i^k \Delta^{\mathtt{d}} + z\left(q_j,\Delta^{\mathtt{d}} N\right)^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k N_i \Delta^{\mathtt{d}} g\left(\frac{1}{\Delta^{\mathtt{d}}}\frac{k^{\mathtt{d}}}{N_i}\right)\right\} + z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta \\
&= \sup_{\{n_i\geq 0\}} k^{\mathtt{d}} \sum_{i\in I^k} \left\{-n_i \underline{R}_i^k + z\left(q_j, k^{\mathtt{d}} \sum_{\tilde{i}\in I^k} n_{\tilde{i}}\right)^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k n_i g\left(\frac{1}{n_i}\right)\right\} + z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta
\end{aligned}
$$

Taking the limit gives

$$\begin{aligned}
\lim_{\Delta\to 0}\bar{\pi}_j^{k\Delta} &= \lim_{\Delta\to 0}\sup_{\{n_i\geq 0\}} k^{\mathtt{d}}\sum_i\left\{-n_i\underline{R}_i^k + z\left(q_j, k^{\mathtt{d}}\sum_{\tilde{i}} n_{\tilde{i}}\right)^{\varepsilon-1}\bar{D}_i^k\bar{b}_i^k n_i g\left(\frac{1}{n_i}\right)\right\} + \\
&+ \lim_{\Delta\to 0} z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta \\
&= \sup_{\{n_i\geq 0\}} k^{\mathtt{d}}\sum_i\left\{-n_i\underline{R}_i^k + z\left(q_j, k^{\mathtt{d}}\sum_{\tilde{i}} n_{\tilde{i}}\right)^{\varepsilon-1}\bar{D}_i^k\bar{b}_i^k n_i g\left(\frac{1}{n_i}\right)\right\} \\
&= \sup_{\{n_i\geq 0\}} k^{\mathtt{d}}\sum_i\left\{-n_i\underline{R}_i^k + z\left(q_j, k^{\mathtt{d}}\sum_{\tilde{i}} n_{\tilde{i}}\right)^{\varepsilon-1}\bar{D}_i^k\bar{b}_i^k \kappa\left(n_i\right)\right\}
\end{aligned}$$

where we used the assumption that $t(\delta)$ diverges as $\delta\to\infty$. Let $\mathcal{N}^k$ be the set of strategies in which n_s is constant for all $s\in\mathcal{S}_i^k$. Then we can write

$$\begin{aligned}
\lim_{\Delta\to 0}\bar{\pi}_j^{k\Delta} &= \sup_{n\in\mathcal{N}^k}\int\left\{-n_s\underline{R}_s^k + z\left(q_j, \int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}\bar{D}_s^k\bar{b}_s^k\kappa\left(n_s\right)\right\} ds \\
&\leq \sup_{n\geq 0}\int\left\{-n_s\underline{R}_s^k + z\left(q_j, \int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}\bar{D}_s^k\bar{b}_s^k\kappa\left(n_s\right)\right\} ds \\
&= \bar{\pi}^k
\end{aligned}$$

■

We next show that for each j, it is without loss of generality to impose a uniform upper bound on the density of plants.

Lemma A.13 *Define $\bar{n}_j$ to satisfy $\underline{R} = z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}\kappa'(\bar{n}_j)$.*

$$\bar{\pi}_j^k = \sup_{n\in[0,\bar{n}_j]}\int\left\{-n_s\underline{R}_s^k + z\left(q_j, \int_{\tilde{s}} n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}\bar{D}_s^k\bar{b}_s^k\kappa\left(n_s\right)\right\} ds$$

Proof. We show that it is without loss to restrict the strategies to the set $n\in[0,\bar{n}_j]$. Restricting the set of strategies yields a weakly lower payoff. To show the opposite inequality, for any strategy, let N^+ be the subset of $\mathcal{S}$ for which $n_s>\bar{n}_j$. We will show that the alternative strategy in which

$$\tilde{n} = \begin{cases} n_s & s\notin N^+ \\ \bar{n}_j & s\in N^+ \end{cases}$$

would give a weakly higher payoff, Consider any profile R, D, b such that $R_s\geq\underline{R}$ and $D_s\leq\bar{D}$ and $b_s\leq\bar{b}$.

For shorthand, we express $z_j = z\left(q_j, \int_s n_s ds\right)$ and $\tilde{z}_j = z\left(q_j, \int_s \tilde{n}_s ds\right)$.

$$\begin{aligned}
\Pi_j(n) - \Pi_j(\tilde{n}) &= \int \left\{-n_s R_s + z_j^{\varepsilon-1} D_s b_s h\kappa(n_s)\right\} ds - \Pi(\tilde{n}) \\
&\leq \int \left\{-n_s R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \kappa(n_s)\right\} ds - \Pi(\tilde{n}) \\
&= \int_{s\in N^+} \left\{-n_s R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \kappa(n_s)\right\} ds - \int_{s\in N^+} \left\{-\tilde{n}_s R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \kappa(\tilde{n}_s)\right\} ds \\
&= \int_{s\in N^+} \left\{-n_s R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \kappa(n_s)\right\} ds - \int_{s\in N^+} \left\{-\bar{n}_s R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \kappa_s(\bar{n})\right\} ds \\
&= \int_{s\in N^+} \left\{-(n_s - \bar{n}_j) R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \left[\kappa(n_s) - \kappa(\bar{n})\right]\right\} ds
\end{aligned}$$

The concavity of κ implies $\kappa(n_s) \leq \kappa(\bar{n}_j) + \kappa'(\bar{n}_j)(n_s - \bar{n}_j)$, so that $\kappa(n_s) - \kappa(\bar{n}_j) \leq \kappa'(\bar{n}_j)(n_s - \bar{n}_j) = \frac{\underline{R}}{z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}}(n_s - \bar{n}_j)$. Plugging this in gives

$$\begin{aligned}
\Pi_j(n) - \Pi_j(\tilde{n}) &\leq \int_{s\in N^+} \left\{-(n_s - \bar{n}_j) R_s + \tilde{z}_j^{\varepsilon-1} D_s b_s \frac{\underline{R}}{z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}}(n_s - \bar{n}_j)\right\} ds \\
&= \int_{s\in N^+} \left\{-1 + \frac{\tilde{z}_j^{\varepsilon-1} D_s b_s}{z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}} \frac{\underline{R}}{R_s}\right\} R_s (n_s - \bar{n}_j)\, ds \\
&\leq 0
\end{aligned}$$

■

Claim A.14

$$\pi_j \leq \sup_{n\geq 0} \int \left\{-n_s R_s + z\left(q_j, \int_{\tilde{s}} n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1} D_s b_s \kappa(n_s)\right\} ds$$

Proof. For any strategy n, define $\bar{\Pi}^k(n)$ and $\Pi(n)$ as

$$\begin{aligned}
\bar{\Pi}^k(n) &= \int \left\{-n_s \underline{R}_s^k + z\left(q, \int_{\tilde{s}} n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1} \bar{D}_s^k \bar{b}_s^k \kappa(n_s)\right\} ds \\
\Pi(n) &= \int \left\{-n_s R_s + z\left(q_j, \int_{\tilde{s}} n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1} D_s b_s \kappa(n_s)\right\} ds
\end{aligned}$$

Since R, D, and b are continuous on a compact space, they are uniformly continuous. This implies that for any $\varphi > 0$, there is an η small enough so that $k < \eta$ implies both $\left|\underline{R}_s^k - R_s\right| \leq \varphi$, and $\left|\bar{D}_s^k \bar{b}_s^k - D_s b_s\right| \leq \varphi$.

With that, for any $n \in [0, \bar{n}_j]$,

$$\begin{aligned}
\left|\bar{\Pi}^k(n) - \Pi(n)\right| &= \left|\int \left\{-n_s\left(\underline{R}_s^k - R_s\right) + z\left(q_j, \int_{\tilde{s}} n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1}\left[\bar{D}_s^k \bar{b}_s^k - D_s b_s\right]\kappa\left(n_s\right)\right\} ds\right| \\
&\leq \int \left\{n_s\left|\underline{R}_s^k - R_s\right| + z\left(q_j, \int_{\tilde{s}} n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1}\left|\bar{D}_s^k \bar{b}_s^k - D_s b_s\right|\kappa\left(n_s\right)\right\} ds \\
&\leq \int \left\{n_s\varphi + z\left(q_j, \int_{\tilde{s}} n_{\tilde{s}} d\tilde{s}\right)^{\varepsilon-1}\varphi\kappa\left(n_s\right)\right\} ds \\
&\leq \int \left\{\bar{n}_j\varphi + z\left(q_j, 0\right)^{\varepsilon-1}\varphi\right\} ds \\
&\leq \varphi\int \left\{\bar{n}_j + z\left(q_j, 0\right)^{\varepsilon-1}\right\} ds
\end{aligned}$$

Therefore $\bar{\Pi}^k(\cdot)$ is uniformly convergent on the domain $n \in [0, \bar{n}_j]$ as $k \to 0$. Therefore

$$\lim_{k\to 0} \sup_{n\in[0,\bar{n}_j]} \bar{\Pi}^k(n) = \sup_{n\in[0,\bar{n}_j]} \lim_{k\to 0} \bar{\Pi}^k(n)$$

In other words, we have $\pi_j \leq \bar{\pi}_j^k$ for all k, so taking the limit of both sides yields

$$\pi_j \leq \lim_{k\to 0} \bar{\pi}_j^k = \lim_{k\to 0} \sup_{n\in[0,\bar{n}_j]} \bar{\Pi}^k(n) = \sup_{n\in[0,\bar{n}_j]} \lim_{k\to 0} \bar{\Pi}^k(n) = \sup_{n\in[0,\bar{n}_j]} \Pi\left(n\right) \leq \sup_{n\geq 0} \Pi(n)$$

■

We next bound the payoff from below.

Claim A.15

$$\pi_j^{\Delta} \geq \underline{\pi}_j^{k\Delta} \equiv \sup_{\{N_i\in\mathbb{N}_0\}} \sum_i -N_i \bar{R}_i^k \xi + Z\left(q_j, \sum_{\tilde{\imath}} N_{\tilde{\imath}}\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k N_i G\left(\rho\left(N_i\right)\frac{k^{\mathtt{d}}}{N_i}\right)$$

Proof. Begin with

$$\begin{aligned}
\int_s D_s \max_{o\in O}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds &= \sum_i \int_{s\in\mathcal{S}_i^k} D_s \max_{o\in O}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds \\
&\geq \sum_i \int_{s\in\mathcal{S}_i^k} D_s \max_{o\in O_i^k}\left\{b_o T\left(\delta_{so}\right)^{1-\varepsilon}\right\} ds \\
&\geq \sum_i \underline{D}_i^k \underline{b}_i^k \int_{s\in\mathcal{S}_i^k} \max_{o\in O_i^k} T\left(\delta_{so}\right)^{1-\varepsilon} ds
\end{aligned}$$

Similarly,

$$\sum_{o\in O}-R_o\xi=\sum_i\sum_{o\in O_i^k}-R_o\xi\geq\sum_i\sum_{o\in O_i^k}-\bar{R}_i^k\xi=\sum_i-\left|O_i^k\right|\bar{R}_i^k\xi$$

Together, these yield a lower bound for π_j^Δ

$$\begin{aligned}
\pi_j^\Delta &= \sup_O\sum_{o\in O}-R_o\xi+Z\left(q_j,|O|\right)^{\varepsilon-1}\int_s D_s\max_{o\in O}\left\{b_oT\left(\delta_{so}\right)^{1-\varepsilon}\right\}ds\\
&\geq \sup_O\sum_i-\left|O_i^k\right|\bar{R}_i^k\xi+Z\left(q_j,|O|\right)^{\varepsilon-1}\sum_i\underline{D}_i^k\underline{b}_i^k\int_{s\in\mathcal{S}_i^k}\max_{o\in O_i}T\left(\delta_{so}\right)^{1-\varepsilon}ds\\
&= \sup_{\{N_i\in\mathbb{N}_0\}}\sup_{\left\{O_i^k\subset\mathcal{S}_i^k|\left|O_i^k\right|=N_i\right\}}\sum_i-\left|O_i^k\right|\bar{R}_i^k\xi+Z\left(q_j,|O|\right)^{\varepsilon-1}\sum_i\underline{D}_i^k\underline{b}_i^k\int_{s\in\mathcal{S}_i^k}\max_{o\in O_i^k}T\left(\delta_{so}\right)^{1-\varepsilon}ds\\
&= \sup_{\{N_i\in\mathbb{N}_0\}}\sup_{\left\{O_i\subset\mathcal{S}_i^k|\left|O_i^k\right|=N_i\right\}}\sum_i-N_i\bar{R}_i^k\xi+Z\left(q,\sum_i N_i\right)^{\varepsilon-1}\sum_i\underline{D}_i^k\underline{b}_i^k\int_{s\in\mathcal{S}_i^k}\max_{o\in O_i^k}T\left(\delta_{so}\right)^{1-\varepsilon}ds\\
&= \sup_{\{N_i\in\mathbb{N}_0\}}\sum_i-N_i\bar{R}_i^k\xi+Z\left(q_j,\sum_i N_i\right)^{\varepsilon-1}\sum_i\underline{D}_i^k\underline{b}_i^k\sup_{\left\{O_i^k\subset\mathcal{S}_i^k|\left|O_i^k\right|=N_i\right\}}\int_{s\in\mathcal{S}_i^k}\max_{o\in O_i^k}T\left(\delta_{so}\right)^{1-\varepsilon}ds\\
&\geq \sup_{\{N_i\in\mathbb{N}_0\}}\sum_i-N_i\bar{R}_i^k\xi+Z\left(q_j,\sum_i N_i\right)^{\varepsilon-1}\sum_i\underline{D}_i^k\underline{b}_i^k N_iG\left(\rho\left(N_i\right)\frac{k^{\mathtt{d}}}{N_i}\right)\\
&= \underline{\pi}_j^{k\Delta}
\end{aligned}$$

■

Claim A.16 *For any k, in the limit as $\Delta\to 0$,*

$$\lim_{\Delta\to0}\underline{\pi}_j^{k\Delta}\geq\underline{\pi}^k\equiv\sup_{\{n_s\geq0\}}\int_s\left\{-n_s\bar{R}_s^k+z\left(q_j,\int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}\underline{D}_s^k\underline{b}_s^k\kappa\left(n_s\right)\right\}ds$$

Proof. Replace and use $n_i = \frac{\Delta^{\mathtt{d}} N_i}{k^{\mathtt{d}}}$

$$\begin{aligned}
\underline{\pi}_j^{k\Delta} &\equiv \sup_{\{N_i \in \mathbb{N}_0\}} \sum_i -N_i \bar{R}_i^k \xi + Z\left(q_j, \sum_i N_i\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k N_i G\left(\rho(N_i)\frac{k^{\mathtt{d}}}{N_i}\right) \\
&= \sup_{\{N_i \geq 0\}} \sum_i -\lceil N_i \rceil \bar{R}_i^k \xi + Z\left(q_j, \sum_i \lceil N_i \rceil\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k \lceil N_i \rceil G\left(\rho\left(\lceil N_i \rceil\right)\frac{k^{\mathtt{d}}}{\lceil N_i \rceil}\right) \\
&= \sup_{\{N_i \geq 0\}} \sum_i -\lceil N_i \rceil \bar{R}_i^k \Delta^{\mathtt{d}} + z\left(q_j, \Delta^{\mathtt{d}} \sum_i \lceil N_i \rceil\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k \lceil N_i \rceil \Delta^d dg\left(\frac{1}{\Delta^d d}\rho\left(\lceil N_i \rceil\right)\frac{k^{\mathtt{d}}}{\lceil N_i \rceil}\right) \\
&= \sup_{\{n_i \geq 0\}} \sum_i -\left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil \bar{R}_i^k \Delta^{\mathtt{d}} + z\left(q_j, \Delta^{\mathtt{d}} \sum_i \left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k \left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil \Delta^{\mathtt{d}} g\left(\frac{1}{\Delta^{\mathtt{d}}}\rho\left(\left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil\right)\frac{k^{\mathtt{d}}}{\left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil}\right)
\end{aligned}$$

Next, we use the fact that $\liminf_{\Delta \to 0} \sup_{\{n \geq 0\}} f(n, \Delta) \geq \sup_{\{n \geq 0\}} \liminf_{\Delta \to 0} f(n, \Delta)$[42] along with $\lim_{\Delta \to 0} \Delta^{\mathtt{d}} \left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil = k^{\mathtt{d}} n_i$ and $\lim_{u \to \infty} \rho(u) = 1$ to get

$$\begin{aligned}
\lim_{\Delta \to 0} \underline{\pi}_j^{k\Delta} &\geq \sup_{\{n_i \geq 0\}} \lim_{\Delta \to 0} \sum_i -\left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil \bar{R}_i^k \Delta^{\mathtt{d}} + \\
&+ z\left(q_j, \Delta^{\mathtt{d}} \sum_i \left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k \left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil \Delta^{\mathtt{d}} g\left(\frac{1}{\Delta^{\mathtt{d}}}\rho\left(\left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil\right)\frac{k^{\mathtt{d}}}{\left\lceil \frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i \right\rceil}\right) \\
&= \sup_{\{n_i \geq 0\}} \sum_i -k^{\mathtt{d}} n_i \bar{R}_i^k + z\left(q_j, \sum_i k^{\mathtt{d}} n_i\right)^{\varepsilon-1} \sum_i \underline{D}_i^k \underline{b}_i^k k^{\mathtt{d}} n_i g\left(\frac{1}{n_i}\right) \\
&= \sup_{\{n_i \geq 0\}} k^{\mathtt{d}} \sum_i \left\{-n_i \bar{R}_i^k + z\left(q_j, k^{\mathtt{d}} \sum_{\tilde{i}} n_{\tilde{i}}\right)^{\varepsilon-1} \underline{D}_i^k \underline{b}_i^k \kappa(n_i)\right\}
\end{aligned}$$

Since $\kappa(n)$ is strictly concave, Jensen's inequality implies that

$$\sup_{n_s} \int_{s \in \mathcal{S}_i^k} \kappa(n_s)\, ds \text{ subject to } \int_{s \in \mathcal{S}_i^k} n_s ds \leq n_i$$

[42] Quick proof: For any n_0, Δ we have $f(n_0, \Delta) \leq \sup_n f(n, \Delta)$. Taking limits preserves inequalities, so that $\liminf_{\Delta \to 0} f(n_0, \Delta) \leq \liminf_{\Delta \to 0} \sup_n f(n, \Delta)$. The conclusion follows from taking sup of both sides with respect to n_0.

is maximized for $n_s = \frac{n_i}{|\mathcal{S}_i^k|}$, i.e., n_s is constant. This means

$$\begin{aligned}\lim_{\Delta\to 0}\underline{\pi}_j^{k\Delta} &\geq \sup_{\{n_i\geq 0\}} k^{\mathtt{d}}\sum_i\left\{-n_i\bar{R}_i^k + z\left(q_j, k^{\mathtt{d}}\sum_{\tilde{i}} n_{\tilde{i}}\right)^{\varepsilon-1}\underline{D}_i^k\underline{b}_i^k\kappa(n_i)\right\}\\ &= \sup_{\{n_s\geq 0\}}\int_s\left\{-n_s\bar{R}_s^k + z\left(q_j, \int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}\underline{D}_s^k\underline{b}_s^k\kappa(n_s)\right\}ds\\ &= \underline{\pi}_j^k\end{aligned}$$

■

Claim A.17

$$\pi_j \geq \sup_{\{n_i\geq 0\}}\int\left\{-n_sR_s + z\left(q_j, \int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}D_sb_s\kappa(n_s)\right\}ds$$

Proof. We can again use the fact that $\liminf_{h\to 0}\sup_{\{n\geq 0\}} f(n,h) \geq \sup_{\{n\geq 0\}}\liminf_{h\to 0} f(n,h)$ to write as $k\to 0$

$$\begin{aligned}\pi_j &\geq \liminf_{k\to 0}\pi_j^{k\Delta} \geq \liminf_{k\to 0}\underline{\pi}_j^k\\ &\geq \sup_{\{n_s\geq 0\}}\liminf_{k\to 0}\int_s\left\{-n_s\bar{R}_s^k + z\left(q_j, \int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}\underline{D}_s^k\underline{b}_s^k\kappa(n_s)\right\}ds\\ &= \sup_{\{n_s\geq 0\}}\int_s\left\{-n_sR_s + z\left(q_j, \int n_{\tilde{s}}d\tilde{s}\right)^{\varepsilon-1}D_sb_s\kappa(n_s)\right\}ds\end{aligned}$$

■

A.3 Proof of Proposition 2: Convergence of the Policy Function

In this appendix we show uniform convergence of the policy function. We do this in two steps. First, we derive properties of the limiting economy. We show that if the limiting problem has a unique solution, n^*, then for any $\varepsilon > 0$ there exists an $\eta > 0$ such that $n \in \bar{\mathcal{N}}$ and $|\Pi(n) - \Pi(n^*)| < \eta$ imply $\int_{s\in\mathcal{S}}|n_s - n_s^*|\,ds < \varepsilon$, where $\bar{\mathcal{N}}$ is a space of functions with a uniform bound.

In the second step, we study the sequence of economies as $\Delta \to 0$. As in the proof of convergence of the value function in Appendix A.2, we construct a sequence of bounds on the profit function that get tighter as $\Delta \to 0$. We show that for economy Δ, the optimal choice $O^{\Delta*}$ has a corresponding strategy in the limiting economy, $n^{\Delta*}$. As $\Delta \to 0$, the bounds get tighter and two things happen. First, $O^{\Delta*}$ gets close to $n^{\Delta*}$ in the sense that over any Jordan measurable set $\mathcal{A}$, $\Delta^d\left|O^{\Delta*}\cap\mathcal{A}\right|$ uniformly approaches $\int_{s\in\mathcal{A}} n_s^{\Delta*}ds$. Second, the corresponding strategy $n^{\Delta*}$ delivers a value in the limiting economy close to optimum. This, along with the first step, implies that $n^{\Delta*}$ converges to n^*. Namely, we have uniform convergence of the policy function to n^*.

As in the proof of Proposition 1, we use device of using $k \times k$ squares to find upper and lower bounds. In that proof, the key step was to take the limit as $\Delta \to 0$ for a given k and then take $k \to 0$. Here, the key trick is to choose use a sequence of $k = K(\Delta)$, so that as we take the limit as $\Delta \to 0$, the sequence $k = K(\Delta)$ also converges to zero (albeit more slowly than does Δ).

A.3.1 Step 1: Properties of the Limiting Problem

In this section, we show that if there is a unique solution to the limiting problem, then for any $\epsilon > 0$ there exists an $\eta > 0$ such that $|\Pi(n) - \Pi(n^*)| < \eta$ implies $\int_{s\in\mathcal{S}} |n - n_s^*|\, ds < \epsilon$. If the $\bar{\mathcal{N}}$ were compact, then we could use a short, standard proof following Lemma 3.7 of Lucas and Stokey. But we have no reason to believe that $\bar{\mathcal{N}}$ compact. Fortunately, we can use direct methods, and the fact that κ is strictly concave is sufficient.

We beginning by defining several terms.

Let $\mathcal{N} \equiv \{n : \mathcal{S} \to \mathbb{R}\}$ be the set of feasible policies.

Let $\mathcal{N}(N) \equiv \left\{n : \mathcal{S} \to \mathbb{R} \text{ such that } \int_{s\in\mathcal{S}} n_s ds = N\right\}$ be the set of feasible policies for which the firm sets up a measure N of plants.

It will be useful to define a finite upper bound for the measure of plants in a location. Define $\bar{n}$ to satisfy $\kappa'(\bar{n}) = \frac{1}{2}\frac{\underline{R}}{z(q,0)^{\varepsilon-1}\bar{x}}$, where $\bar{x} \equiv (\max_{s\in\mathcal{S}} b_s)(\max_{s\in\mathcal{S}} D_s)$ and $\underline{R} = \min_{s\in\mathcal{S}} R_s$.

Define $\bar{N} \equiv \bar{n}|\mathcal{S}|$ to be an upper bound on the total mass of plants given the upper bounds $\bar{n}$ for any particular location.

Define $\bar{\mathcal{N}} \equiv \{n : \mathcal{S} \to [0, \bar{n}]\}$ to be the set of strategies for which n_s is bounded between 0 and $\bar{n}$, and let $\bar{\mathcal{N}}(N) \equiv \left\{n : \mathcal{S} \to [0, \bar{n}] \text{ such that } \int_{s\in\mathcal{S}} n_s ds = N\right\}$ be the subset of those where the total measure of plants is N.

It will also be useful to have notation for the partial inverse of κ'. $\kappa'(\cdot)$ is strictly decreasing when restricted to a strictly positive domain. Let χ be the partial inverse of κ', so that $\chi(x) \equiv \begin{cases} \kappa'^{-1}(x) & x < \kappa'(0) \\ 0 & x \geq \kappa'(0) \end{cases}$ and note that $\chi(x)$ is continuous.

Lemma A.18 *For any N there exists a unique solution to the problem $\sup_{n\in\mathcal{N}(N)} \Pi(n)$. The optimum $\hat{n}(N)$ and the multiplier $\lambda(N)$ associated with the constraint $\int_{s\in\mathcal{S}} n_s ds = N$ are both continuous in N.*

Proof. Fix N. Consider the problem

$$\max_{n\in\mathcal{N}(N)} \Pi(n) \equiv \max_{n\in\mathcal{N}(N)} \int_{s\in\mathcal{S}} \left[-R_s n_s + x_s z(q, N)^{\varepsilon-1}\kappa(n_s)\right] ds$$

The objective function is strictly concave and the constraint set is convex, so the first order conditions are necessary sufficient to characterize a solution. Letting λ be the multiplier on the constraint $\int_{s\in\mathcal{S}} n_s ds = N$, the first order condition for n_s is

$$R_s + \lambda \geq x_s z^{\varepsilon-1}\kappa'(n_s) \text{ with equality if } n_s > 0.$$

Then the optimal policy and Lagrange multiplier satisfy

$$\begin{aligned}\hat{n}_s(N) &= \chi\left(\frac{R_s+\lambda\,(N)}{x_s z^{\varepsilon-1}}\right)\\ N &= \int_{s\in\mathcal{S}}\chi\left(\frac{R_s+\lambda(N)}{x_s z^{\varepsilon-1}}\right)ds\end{aligned}$$

Note that for any $N>0$, there is a unique $\lambda(N)$ that satisfies the second equation. The continuity of χ thus implies the continuity $\hat{n}_s(N)$ and $\lambda(N)$ in N. ■

Lemma A.19 *Suppose there is a unique solution, n^*, and that $N^*=\int_{s\in\mathcal{S}} n_s^* ds$. Then*

$$\lim_{N\to N^*}\int_{s\in\mathcal{S}}\left|\hat{n}_s(N)-n_s^*\right|ds=0.$$

Proof. Since n^* is optimal, $\lambda(N^*)=-\frac{d\left[z(q,N^*)^{\varepsilon-1}\right]}{dN^*}\int_{s\in\mathcal{S}} x_s\kappa(n_s^*)ds\geq 0$. $\lambda(N)$ is continuous, so that $\lambda(N)>-\underline{R}/2$ in a neighborhood of N^*. As a result, the first order condition that $R_s+\lambda(N)\geq z(q,N)^{\varepsilon-1}x_s\kappa'(\hat{n}_s(N))$ with equality if $\hat{n}_s(N)>0$ means that we can find a positive lower bound for $\kappa'(\hat{n}_s(N))$ whenever $\lambda(N)>0$. Either $\hat{n}_s(N)=0$ or

$$\kappa'(\hat{n}_s(N))=\frac{R_s+\lambda(N)}{x_s z(q,N)^{\varepsilon-1}}\geq\frac{\underline{R}/2}{\bar{x}z(q,0)^{\varepsilon-1}}.$$

Define $\mathring{n}$ to satisfy $\kappa'(\mathring{n})=\frac{\underline{R}/2}{\bar{x}z(q,0)^{\varepsilon-1}}$. Then $\mathring{n}_s\geq n_s^*$ and, if N is sufficiently close to N^*, $\mathring{n}_s>\hat{n}_s(N)$. Then $|n_s^*-\hat{n}_s(N)|\leq 2\mathring{n}$ when N is close enough to N^*. $2\mathring{n}$ is intregrable over $\mathcal{S}$, and $\hat{n}(N)$ converges to n pointwise, so dominated convergence implies that $\lim_{N\to N^*}\int_{s\in\mathcal{S}}\left|\hat{n}_s(N)-n_s^*\right|ds=0$. ■

Lemma A.20 *The maximization problem $\sup_{n\in\mathcal{N}}\Pi(n)$ obtains a maximum, Π^*. If $n\in\arg\max_{n\in\mathcal{N}}\Pi(n)$ then $\int n_s ds\leq\bar{N}$. If $\int_{s\in\mathcal{S}} n_s ds>\bar{N}$, then $\Pi^*-\Pi(n)\geq\frac{1}{2}\underline{R}(\int_{s\in\mathcal{S}} n_s ds-\bar{N})$.*

Proof. $[0,\bar{N}]$ is a closed, bounded segment of the real line, so it is compact. Further $\Pi\left(\hat{n}(N)\right)$ is continuous in N. Thus $\Pi\left(\hat{n}(N)\right)$ obtains the maximum on $N\in\left[0,\bar{N}\right]$.

We next show that any strategy n such that $\int n_s ds>\bar{N}$ is strictly dominated. Let $\tilde{n}$ be defined so that $\tilde{n}_s=\min\left\{n_s,\bar{n}\right\}$. Letting $N\equiv\int_{s\in\mathcal{S}} n_s ds$ and $\tilde{N}\equiv\int_{s\in\mathcal{S}}\tilde{n}_s ds$, note that $z(q,\tilde{N})^{\varepsilon-1}\geq z(q,N)^{\varepsilon-1}$. This implies

$$\begin{aligned}\Pi\left(\tilde{n}\right)-\Pi(n) &= \int\left\{-R_s\tilde{n}_s+z\left(q,\tilde{N}\right)^{\varepsilon-1}x_s\kappa(\tilde{n}_s)\right\}ds-\int\left\{-R_s n_s+z\left(q,N\right)^{\varepsilon-1}x_s\kappa(n_s)\right\}ds\\ &\geq \int\left\{-R_s\tilde{n}_s+z\left(q,\tilde{N}\right)^{\varepsilon-1}x_s\kappa\left(\tilde{n}_s\right)\right\}ds-\int\left\{-R_s n_s+z\left(q,\tilde{N}\right)^{\varepsilon-1}x_s\kappa\left(n_s\right)\right\}ds\\ &= \int\left\{-R_s\left(\tilde{n}_s-n_s\right)+z\left(q,\tilde{N}\right)^{\varepsilon-1}x_s\left[\kappa(\tilde{n}_s)-\kappa(n_s)\right]\right\}ds\end{aligned}$$

Since $\kappa(n_s)-\kappa(\tilde{n}_s)\leq\kappa'(\bar{n})(n_s-\tilde{n}_s)=\frac{1}{2}\frac{\underline{R}}{z(q,0)^{\varepsilon-1}\bar{x}}(n_s-\tilde{n}_s)$, this is

$$\begin{aligned}
\Pi(\tilde{n})-\Pi(n) &\geq \int\left\{-R_s(\tilde{n}_s-n_s)+\frac{1}{2}\frac{z\left(q,\tilde{N}\right)^{\varepsilon-1}}{z(q,0)^{\varepsilon-1}}\frac{x_s}{\bar{x}}\underline{R}(\tilde{n}_s-n_s)\right\}ds \\
&= \int\left[\frac{R_s}{\underline{R}}-\frac{1}{2}\frac{z\left(q,\tilde{N}\right)^{\varepsilon-1}}{z(q,0)^{\varepsilon-1}}\frac{x_s}{\bar{x}}\right]\underline{R}(n_s-\tilde{n}_s)\,ds \\
&\geq \int\frac{1}{2}\underline{R}(n_s-\tilde{n}_s)\,ds \\
&\geq \int\frac{1}{2}\underline{R}(n_s-\bar{n})\,ds \\
&= \frac{1}{2}\underline{R}\left(\int n_s ds-N\right)
\end{aligned}$$

The conclusion follows from $\int n_s ds>N$ and $Pi^*\geq\Pi(\tilde{n})$. ■

Lemma A.21 *Suppose that there is a unique solution n^*. For any $\gamma>0$, there is an $\eta_1>0$ such that for all $n\in\mathcal{N}$, $|\Pi(n)-\Pi(n^*)|<\eta_1$ implies $\left|\int_{s\in\mathcal{S}}n_s ds-\int_{s\in\mathcal{S}}n_s^* ds\right|<\gamma$.*

Proof. We first consider functions n such that $\int_{s\in\mathcal{S}}n_s ds\in[0,\bar{N}+\gamma]$. Let $N^*=\int_{s\in\mathcal{S}}n_s^* ds$. For $\gamma>0$, define $E_\gamma=\left\{N\in\left[0,\bar{N}+\gamma\right] \text{ such that } |N-N^*|\geq\gamma\right\}$. Lemma A.20 states that $N^*\in[0,\bar{N}]$, so that E_γ is non-empty and compact. For any such γ, let $\eta\equiv\min_{N\in E_\gamma}\left|\max_{n\in\mathcal{N}(N)}\Pi(n)-\Pi(n^*)\right|$. Since the function being minimized is continuous in N and E_γ is compact, the minimum is attained. Moreover, since $N^*\notin E_\gamma$, it follows that $\eta>0$. As a result, any $N\in[0,\bar{N}+\gamma]$ with $|N-N^*|\geq\gamma$ implies that $\left|\max_{n\in\mathcal{N}(N)}\Pi(n)-\Pi(n^*)\right|\geq\eta$.

We next consider functions n is such that $\int_{s\in\mathcal{S}}n_s ds>\bar{N}+\gamma$. For such functions, Lemma A.20 implies that $\Pi(n^*)-\Pi(n)\geq\frac{1}{2}\underline{R}\gamma$.

Together, we have that if $|\Pi(n^*)-\Pi(n)|\leq\eta_1\equiv\min\left\{\eta,\frac{1}{2}\underline{R}\gamma\right\}$ then $\left|\int_{s\in\mathcal{S}}n_s ds-\int_{s\in\mathcal{S}}n_s^* ds\right|<\gamma$.

■

Lemma A.22 *Let $\hat{n}(N)\equiv\arg\max_{n\in\mathcal{N}(N)}\Pi(n)$. For any $\epsilon>0$, there exists a function $h_\epsilon(N)$ that is continuous and strictly positive on $N\in(0,\infty)$, such that for any $\tilde{n}\in\mathcal{N}(N)$, $\Pi(\hat{n}(N))-\Pi(\tilde{n})\geq h_\epsilon(N)$ implies $\|\hat{n}(N)-\tilde{n}\|>\epsilon$.*

Proof. Fix $\epsilon>0$ and $N>0$. We will omit the argument N whenever there is no ambiguity.

Let $\lambda(N)$ be the multiplier on the constraint $\int n_s ds=N$ in the maximization problem.

Define $\omega_s(u)\equiv-R_s u-\lambda u+x_s z^{\varepsilon-1}\kappa(u)$. For any $s\in\mathcal{S}$ and $\tau\in(0,\infty)$, $\omega_s(\hat{n}_s)-\omega_s(\hat{n}_s+\tau)$ is strictly positive and strictly increasing in τ. To see this note the fact that $\hat{n}$ is optimal implies that $\omega_s'(\hat{n}_s)\leq 0$ (with equality if $\hat{n}_s>0$). Further, ω_s is strictly concave because it inherits the strict concavity of κ. Thus for any

$\tau > 0$ the strict concavity implies that $\omega_s(\hat{n}_s) - \omega_s(\hat{n}_s + \tau)$ is strictly positive, $\omega_s(\hat{n}_s + \tau) < \omega_s(\hat{n}_s) + \tau\omega_s'(\hat{n}_s) \leq \omega_s(\hat{n}_s)$, and also that it is strictly increasing:

$$\frac{d}{d\tau}\left[\omega_s(\hat{n}_s) - \omega_s(\hat{n}_s + \tau)\right] = -\omega_s'(\hat{n}_s + \tau) > -\omega_s'(\hat{n}_s) \geq 0.$$

Next, define the functions

$$\begin{aligned} H_\epsilon(N) &\equiv \inf_{n \in \mathcal{N}(N)} \Pi\left(\hat{n}(N)\right) - \Pi(n) \text{ subject to } \int_{s\in\mathcal{S}} |n_s - \hat{n}_s|\, ds \geq \epsilon \\ h_\epsilon(N) &\equiv \inf_{\{\tau_s \geq 0\}_{s\in\mathcal{S}}} \int_{s\in\mathcal{S}} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(\hat{n}_s + \tau_s\right)\right] ds \text{ subject to } \int_{s\in\mathcal{S}} \tau_s ds = \frac{\epsilon}{2} \end{aligned}$$

We now show that $H_\epsilon(N) \geq h_\epsilon(N)$. For any n such that $\int_{s\in\mathcal{S}} \hat{n}_s ds = \int_{s\in\mathcal{S}} n_s ds$, we can multiply both sides by $\lambda(N)$ and rearrange to get $\int_{s\in\mathcal{S}} \lambda(N)\left(n_s - \hat{n}_s\right) ds = 0$, meaning that we can rearrange $H_\epsilon(N)$ as

$$H_\epsilon(N) = \inf_{n\in\mathcal{N}(N)} \int_{s\in\mathcal{S}} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(n_s\right)\right] ds \text{ subject to } \int_{s\in\mathcal{S}} |n_s - \hat{n}_s|\, ds \geq \epsilon$$

We can rearrange this further as

$$H_\epsilon(N) = \inf_{n\in\mathcal{N}(N)} \int_{s\in\mathcal{S}|n_s<\hat{n}_s} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(n_s\right)\right] ds + \int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(n_s\right)\right] ds$$

subject to $\int_{s\in\mathcal{S}|n_s<\hat{n}_s} |n_s - \hat{n}_s|\, ds = \int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} |n_s - \hat{n}_s|\, ds$ and $\int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} |n_s - \hat{n}_s|\, ds \geq \frac{\epsilon}{2}$. Since $\omega_s\left(\hat{n}_s\right) \geq \omega_s\left(n_s\right)$,we have

$$H_\epsilon(N) \geq \inf_{n\in\mathcal{N}(N)} \int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(n_s\right)\right] ds$$

subject to $\int_{s\in\mathcal{S}|n_s<\hat{n}_s} |n_s - \hat{n}_s|\, ds = \int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} |n_s - \hat{n}_s|\, ds$ and $\int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} |n_s - \hat{n}_s|\, ds \geq \frac{\epsilon}{2}$. Relaxing a constraint gives a weakly smaller number, so that

$$\begin{aligned} H_\epsilon(N) &\geq \inf_{n\in\mathcal{N}(N)} \int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(n_s\right)\right] ds \text{ subject to } \int_{s\in\mathcal{S}|n_s\geq\hat{n}_s} |n_s - \hat{n}_s|\, ds \geq \frac{\epsilon}{2} \\ &\geq \inf_{\{\tau_s\geq 0\}_{s\in\mathcal{S}}} \int_{s\in\mathcal{S}} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(\hat{n}_s + \tau_s\right)\right] ds \text{ subject to } \int_{s\in\mathcal{S}} \tau_s ds \geq \frac{\epsilon}{2} \end{aligned}$$

Since $\omega_s\left(\hat{n}_s\right) - \omega_s\left(\hat{n}_s + \tau\right)$ is increasing and convex in τ on $\tau \in [0, \infty)$, the value of the right hand side is unchanged if we impose that the constraint holds with equality. This gives

$$\begin{aligned} H_\epsilon(N) &\geq \inf_{\{\tau_s\geq 0\}_{s\in\mathcal{S}}} \int_{s\in\mathcal{S}} \left[\omega_s\left(\hat{n}_s\right) - \omega_s\left(\hat{n}_s + \tau_s\right)\right] ds \text{ subject to } \int_{s\in\mathcal{S}} \tau_s ds = \frac{\epsilon}{2} \\ &= h_\epsilon(N) \end{aligned}$$

We next show that $h_\epsilon(N)$ is strictly increasing in ϵ. To do this, we solve the minimization problem. The objective function is strictly convex and the constraint set is convex, so the first order conditions are necessary and sufficient for a minimum. Let μ be the multiplier on the constraint. Then the first order condition for τ_s is

$$\omega_s'\left(\hat{n}_s+\tau_s\right)+\mu\leq 0 \text{ with equality if } \tau_s>0$$

or, equivalently,

$$-R_s-\lambda(N)+x_s z^{\varepsilon-1}\kappa'\left(\hat{n}_s+\tau_s\right)+\mu\leq 0 \quad \text{with equality if} \quad \tau_s>0$$

Since ω' is continuous, τ_s is continuous is s.

Note that it must be that $\mu>0$: otherwise the FOC for τ_s would imply $\tau_s=0$, $\forall s$ because $\omega_s\left(\hat{n}_s+\tau\right)$ is strictly concave on $\tau\in(0,\infty)$ and $\omega_s'\left(\hat{n}_s\right)\leq 0$, $\forall s$, and this would violate the constraint. Note also that $R_s+\lambda>\mu$ for all s: $\lim_{\tau\to\infty}\omega_s\left(n+\tau\right)=-\infty$ implies that the optimal τ is finite, so that the FOC implies $-R_s-\lambda(N)+\mu\leq -x_s z^{\varepsilon-1}\kappa'\left(\hat{n}_s+\tau_s\right)<0$:. Recalling that χ is the partial inverse of κ' and that χ is continuous, the first order condition for τ_s can be restated as

$$\tau_s=\max\left\{0,\chi\left(\frac{R_s+\lambda-\mu}{x_s z^{\varepsilon-1}}\right)-\hat{n}_s\right\}. \tag{18}$$

μ must therefore satisfy

$$\int_{s\in\mathcal{S}}\max\left\{0,\chi\left(\frac{R_s+\lambda-\mu}{x_s z^{\varepsilon-1}}\right)-\hat{n}_s\right\}=\frac{\epsilon}{2}$$

The continuity and monotonicity of χ implies that this has a unique solution, $\mu(\epsilon,N)$ that is continuous in ϵ. Note also that $\mu(\epsilon,N)$ is continuous in N for fixed ϵ.

We next show that $h_\epsilon(N)$ is strictly increasing in ϵ. Consider $\epsilon>0$. Let $\tau_s(\epsilon)$ be the solution for ϵ described by (18). Note that τ_s is continuous in s. There must be an $\bar{\eta}>0$ such that the set $E=\{s|\tau_s\left(\epsilon\right)\geq\bar{\eta}\}$ has strictly positive measure (if no such $\bar{\eta}$, E existed, it would violate the constraint $\int_{s\in\mathcal{S}}\tau_s(\epsilon)ds\geq\frac{\epsilon}{2}$). Since τ_s is continuous and E is bounded, E is compact. Consider ϵ' such that $0\leq\frac{\epsilon}{2}-\bar{\eta}|E|<\frac{\epsilon'}{2}<\frac{\epsilon}{2}$. For any such ϵ', let $\eta=\frac{1}{|E|}\left(\frac{\epsilon}{2}-\frac{\epsilon'}{2}\right)\in(0,\bar{\eta})$. Consider the strategy of $\tau_s(\epsilon')=\tau_s(\epsilon)-\eta$ if $s\in E$ and $\tau_s(\epsilon')=\tau_s(\epsilon)$ if $s\notin E$. Then

$$\begin{aligned}
h_{\epsilon'}(N)-h_\epsilon(N) &\leq \int_{s\in\mathcal{S}}\left[\omega_s\left(\hat{n}_s+\tau_s\left(\epsilon\right)\right)-\omega_s\left(\hat{n}_s+\tau_s\left(\epsilon'\right)\right)\right]ds\\
&= \int_{s\in E}\left[\omega_s\left(\hat{n}_s+\tau_s\left(\epsilon\right)\right)-\omega_s\left(\hat{n}_s+\tau_s\left(\epsilon\right)-\eta\right)\right]ds\\
&\leq |E|\sup_{s\in E}\left[\omega_s\left(\hat{n}_s+\tau_s\left(\epsilon\right)\right)-\omega_s\left(\hat{n}_s+\tau_s\left(\epsilon\right)-\eta\right)\right]
\end{aligned}$$

Since E is compact while $\omega_s\left(\hat{n}_s+\tau_s(\epsilon)\right)-\omega_s\left(\hat{n}_s+\tau_s(\epsilon)-\eta\right)$ is continuous and strictly negative, the supremum is strictly negative, giving $h_{\epsilon'}(N)<h_\epsilon(N)$.

We next establish that for $\epsilon > 0$, $h_\epsilon(N)$ is strictly positive and continuous in N. The fact that $h_\epsilon(N)$ is strictly increasing in ϵ and $h_0(N) = 0$ implies that $h_\epsilon(N)$ is strictly positive when $\epsilon > 0$. Further, continuity in N follows from the continuity of $\hat{n}(N)$, $\tau_s(\epsilon, N)$, $z(q, N)$, $\lambda(N)$, and $\mu(\epsilon, N)$ in N and the continuity of $\omega_s(\hat{n}_s) - \omega_s(\hat{n}_s + \tau)$ in τ. ■

Claim A.23 *Suppose that there is a unique solution, n^*. Then for any $\epsilon > 0$, there is an $\eta > 0$ such that for any $n \in \mathcal{N}$, $|\Pi(n) - \Pi(n^*)| < \eta$ implies $\int_{s\in\mathcal{S}} |n_s - n_s^*|\, ds < \epsilon$.*

Proof. Fix $\epsilon > 0$. Let $N^* \equiv \int_{s\in\mathcal{S}} n_s^* ds$ and let $\hat{n}(N) \equiv \arg\max_{n\in\mathcal{N}(N)} \Pi(n)$. Using Lemma A.19, there is a $\gamma > 0$ such that $|N - N^*| < \gamma$ implies that $\int_{s\in\mathcal{S}} |n_s^* - \hat{n}_s(N)|\, ds < \frac{\epsilon}{2}$. Define

$$J(\epsilon, N) = \inf_{n\in\mathcal{N}(N)} \Pi(n^*) - \Pi(n) \text{ subject to } \int_{s\in\mathcal{S}} |n_s - n_s^*|\, ds \geq \epsilon$$

We will show that there is a strictly-positive, uniform lower bound $\underline{J}(\epsilon)$ on $J(\epsilon, N)$ for any $N \in [N^* - \gamma, N^* + \gamma]$.

Since $\Pi(n^*) \geq \Pi(\hat{n}(N))$, we have

$$J(\epsilon, N) \geq \inf_{n\in\mathcal{N}(N)} \Pi(\hat{n}(N)) - \Pi(n) \text{ subject to } \int_{s\in\mathcal{S}} |n_s - n_s^*| ds \geq \epsilon$$

Relaxing the constraint delivers a lower bound. In particular, since $\int_{s\in\mathcal{S}} |n_s - n_s^*|\, ds \leq \int_{s\in\mathcal{S}} |n_s - \hat{n}_s(N)|\, ds + \int_{s\in\mathcal{S}} |\hat{n}_s(N) - n_s^*|\, ds$, the constraint can be relaxed to $\int_{s\in\mathcal{S}} |n_s - \hat{n}_s(N)|\, ds \geq \epsilon - \int_{s\in\mathcal{S}} |\hat{n}_s(N) - n_s^*|\, ds$, and since $\int_{s\in\mathcal{S}} |\hat{n}_s(N) - n_s^*|\, ds < \frac{\epsilon}{2}$, a further relaxation is $\int_{s\in\mathcal{S}} |n_s - \hat{n}_s(N)|\, ds \geq \frac{\epsilon}{2}$. As a result,

$$J(\epsilon, N) \geq \inf_{n\in\mathcal{N}(N)} \Pi(\hat{n}(N)) - \Pi(n_s) \text{ subject to } \int_{s\in\mathcal{S}} |n_s - \hat{n}_s(N)|\, ds \geq \frac{\epsilon}{2}$$

Using the previous lemma, this implies that $J(\epsilon, N) \geq h_{\epsilon/2}(N)$. Define the uniform lower bound

$$\begin{aligned} \underline{J}(\epsilon) &\equiv \inf_{N\in[N^*-\gamma, N^*+\gamma]} J(\epsilon, N) \\ &\geq \inf_{N\in[N^*-\gamma, N^*+\gamma]} h_{\epsilon/2}(N) \end{aligned}$$

Since $[N^* - \gamma, N^* + \gamma]$ is compact and $h_{\epsilon/2}(N)$ is continuous and strictly positive, the infimum achieves a minimum which is strictly positive, i.e., $\underline{J}(\epsilon) > 0$, which further implies that $J(\epsilon, N) \geq \underline{J}(\epsilon) > 0$, $\forall N \in [N^* - \gamma, N^* + \gamma]$. To summarize, we have established that if $\Pi(n^*) - \Pi(n) < \bar{J}(\epsilon)$ and $\left|\int n_s ds - N^*\right| < \gamma$ then $\int_{s\in S} |n - n_s^*|\, ds < \epsilon$.

According to Lemma A.21, there is an $\eta_1 > 0$ such that $|\Pi(n) - \Pi(n^*)| < \eta_1$ implies that $\left|\int_{s\in\mathcal{S}} n_s ds - N^*\right| < \gamma$. Together, these two results imply that if

$$|\Pi(n) - \Pi(n^*)| < \eta \equiv \min\{\eta_1, \underline{J}(\epsilon)\}$$

then $\int_{s\in S}|n-n_s^*|\,ds<\epsilon$. ■

A.3.2 Step 2: Convergence of the policy function

In this section, we show that if there is a unique solution to the limiting problem, then the appropriately scaled policy function converges uniformly as $\Delta\to 0$.

As in the proof of the Proposition 1, we use the line segments of length k or $k\times k$ squares to derive bounds. In Proposition 1, the key step was to take the limit as $\Delta\to 0$ for a given k and then take $k\to 0$. We cannot use that strategy here, because, if k is held fixed, as we take a limit $\Delta\to 0$, neither the sequence of strategies for the upper bound nor the sequence of strategies for the lower bound converge to n^*. Instead, the key trick here is, as we take $\Delta\to 0$, to let k approach zero as well. Specifically, we define a function the sequence $K(\Delta)$ converges to zero, but more slowly than does Δ in a sense described below. The fact that k approaches zero as well ensures uniform convergence of the (appropriately scaled) policy function.

Let I^k be the set of line segments of length k or squares of size $k\times k$, so that $\mathcal{S}=\bigcap_{i\in I^k}\mathcal{S}_i^k$.

Let $\Pi^\Delta(O)$ be the profit a firm would get in economy Δ if it chooses a set of plants O. For a vector $\mathbf{N}=\{N_i\}_{i\in I^k}$, define

$$U^{k,\Delta}(\mathbf{N})\equiv\sup_O\Pi^\Delta(O)\ \text{ subject to }\ \left|O\cap\mathcal{S}_i^k\right|=N_i$$

to be profit in economy Δ when the policy is constrained so that the firm places N_i plants in $\mathcal{S}_i^k$, $\forall i\in I_k$. In addition, define

$$\hat{n}^{k,\Delta}(\mathbf{N})\equiv\arg\max_n\Pi(n)\qquad\text{subject to }\ \int_{s\in\mathcal{S}_i^k}n_s ds=\Delta^{\mathtt{d}}N_i$$

to be the optimal policy in the limiting economy under the constraints that a measure $\Delta^{\mathtt{d}}N_i$ of plants is placed in $\mathcal{S}_i^k$.

For any k, Δ, and any $\mathbf{N}=\{N_i\}_{i\in I^k}$, define $\bar{U}^{k,\Delta}(\mathbf{N})$ and $\underline{U}^{k,\Delta}(\mathbf{N})$ to be upper and lower bounds on the profit a firm could achieve in economy Δ if it chose to place N_i plants in $\mathcal{S}_i^k$:

$$\begin{aligned}\bar{U}^{k,\Delta}(\mathbf{N}) &\equiv \sum_{i\in I^k}-\underline{R}_i^k\Delta^{\mathtt{d}}N_i+z\left(q,\Delta^{\mathtt{d}}\sum_{\tilde{\imath}\in I^k}N_{\tilde{\imath}}\right)^{\varepsilon-1}\sum_{i\in I^k}\bar{D}_i^k\bar{b}_i^kN_i\Delta^{\mathtt{d}}g\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}N_i}\right)+z(q,0)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}-3}\int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon}dk\\ \underline{U}^{k,\Delta}(\mathbf{N}) &\equiv \sum_{i\in I^k}-\bar{R}_i^k\Delta^{\mathtt{d}}N_i+z\left(q,\Delta^{\mathtt{d}}\sum_{\tilde{\imath}\in I^k}N_{\tilde{\imath}}\right)^{\varepsilon-1}\sum_{i\in I^k}\underline{D}_i^k\underline{b}_i^kN_i\Delta^{\mathtt{d}}g\left(\rho(N_i)\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}N_i}\right)\end{aligned}$$

It follows from the same arguments of Claims A.11 and A.15 that

$$\underline{U}^{k,\Delta}(\mathbf{N})\le U^{k,\Delta}(\mathbf{N})\le\bar{U}^{k,\Delta}(\mathbf{N})$$

Define $K(\Delta)$ to be an increasing function that is increasing more slowly than Δ, so that $\lim_{\Delta\to0}K(\Delta)=0$

but $\lim_{\Delta\to 0} K(\Delta)^{\mathtt{d}-3}\Delta = 0$, and $\frac{1}{K(\Delta)}$ is an integer, e.g., $K(\Delta) = \lceil \Delta^{-1/3} \rceil^{-1}$. One implication is that $\lim_{\Delta\to 0} \frac{\Delta}{K(\Delta)} = \lim_{\Delta\to 0} K(\Delta)^{2-\mathtt{d}} K(\Delta)^{\mathtt{d}-3}\Delta = 0$.

Lemma A.24 *Fix $\epsilon > 0$. There exists a $\bar{\Delta} > 0$ such that for any $\Delta < \bar{\Delta}$ and any $\mathbf{N} = \{N_i\}_{i\in I^{K(\Delta)}}$ such that $\sum_{i\in I^{K(\Delta)}} N_i \leq \frac{2\bar{N}}{\Delta^{\mathtt{d}}}$,*

$$\left|\bar{U}^{K(\Delta),\Delta}(\mathbf{N}) - \underline{U}^{K(\Delta),\Delta}(\mathbf{N})\right| < \epsilon$$

Proof. Fix $\mathbf{N}$, and let $n_i = \frac{\Delta^{\mathtt{d}}}{k^{\mathtt{d}}} N_i$. Then we have

$$\begin{aligned}
\left|\bar{U}^{k,\Delta}(\mathbf{N}) - \underline{U}^{k,\Delta}(\mathbf{N})\right| \leq\ & k^{\mathtt{d}} \sum_i \left\{ -n_i \underline{R}_i^k + z\left(q_j, k^{\mathtt{d}} \sum_i n_i\right)^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k n_i g\left(\frac{1}{n_i}\right) \right\} \\
& + z\left(q_j, 0\right)^{\varepsilon-1} \bar{D}\bar{b} k^{\mathtt{d}-3} \int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta \\
& - k^{\mathtt{d}} \sum_i \left\{ -n_i \bar{R}_i^k + z\left(q_j, k^{\mathtt{d}} \sum_i n_i\right)^{\varepsilon-1} \underline{D}_i^k \underline{b}_i^k n_i g\left(\rho\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i\right) \frac{1}{n_i}\right) \right\}
\end{aligned}$$

This can be rearranged as

$$\left|\bar{U}^{k,\Delta}(\mathbf{N}) - \underline{U}^{k,\Delta}(\mathbf{N})\right| = A_1^{k,\Delta}(\mathbf{N}) + A_2^{k,\Delta}(\mathbf{N}) + A_3^{k,\Delta}(\mathbf{N})$$

where

$$\begin{aligned}
A_1^{k,\Delta}(\mathbf{N}) &\equiv z(q_j, 0)^{\varepsilon-1} \bar{D}\bar{b} k^{\mathtt{d}-3} \int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta \\
A_2^{k,\Delta}(\mathbf{N}) &\equiv k^{\mathtt{d}} \sum_i \left\{ -n_i\left(\underline{R}_i^k - \bar{R}_i^k\right) + z\left(q_j, k^{\mathtt{d}} \sum_{\tilde{\imath}\in I^k} n_{\tilde{\imath}}\right)^{\varepsilon-1} \left(\bar{D}_i^k \bar{b}_i^k - \underline{D}_i^k \underline{b}_i^k\right) n_i g\left(\frac{1}{n_i}\right) \right\} \\
A_3^{k,\Delta}(\mathbf{N}) &\equiv k^{\mathtt{d}} \sum_i \left\{ z\left(q_j, k^{\mathtt{d}} \sum_i n_i\right)^{\varepsilon-1} \underline{D}_i^k \underline{b}_i^k n_i \left[g\left(\frac{1}{n_i}\right) - g\left(\rho\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}} n_i\right) \frac{1}{n_i}\right) \right] \right\}
\end{aligned}$$

We bound each of these three terms separately. First, note that since $\lim_{\Delta\to 0} K(\Delta)^{\mathtt{d}-3}\Delta = 0$, there is a $\bar{\Delta}_1$ small enough so that $\Delta < \bar{\Delta}_1$ implies that $K(\Delta)^{\mathtt{d}-3}\Delta < \frac{1}{z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}\int_0^\infty t\left(\tilde{\delta}\right)^{1-\varepsilon} d\tilde{\delta}} \frac{\epsilon}{3}$. This means that for

$\Delta < \bar{\Delta}_1$:

$$\begin{aligned}
A_1^{K(\Delta),\Delta}(\mathbf{N}) &= z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}K(\Delta)^{\mathtt{d}-3}\int_0^{K(\Delta)} t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta \\
&= z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}K(\Delta)^{\mathtt{d}-3}\Delta\int_0^{\frac{K(\Delta)}{\Delta}} t\left(\tilde{\delta}\right)^{1-\varepsilon} d\tilde{\delta} \\
&< \left(z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}\int_0^{\frac{K(\Delta)}{\Delta}} t\left(\tilde{\delta}\right)^{1-\varepsilon} d\tilde{\delta}\right)\frac{1}{z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}\int_0^{\infty} t\left(\tilde{\delta}\right)^{1-\varepsilon} d\tilde{\delta}}\frac{\epsilon}{3} \\
&= \frac{\int_0^{\frac{K(\Delta)}{\Delta}} t\left(\tilde{\delta}\right)^{1-\varepsilon} d\tilde{\delta}}{\int_0^{\infty} t\left(\tilde{\delta}\right)^{1-\varepsilon} d\tilde{\delta}}\frac{\epsilon}{3} \\
&\leq \frac{\epsilon}{3}
\end{aligned}$$

where the second line used the change of variables $\tilde{\delta} = \frac{\delta}{\Delta}$.

We turn next to the second term, $A_2^{k,\Delta}(\mathbf{N})$. Since R_s, D_s, and b_s are uniformly continuous, there is a $\bar{k}$ such that $k < \bar{k}$ implies $\bar{R}_i^k - \underline{R}_i^k \leq \frac{\epsilon}{6(2\bar{N})}$ and $\bar{D}_i^k\bar{b}_i^k - \underline{D}_i^k\underline{b}_i^k < \frac{1}{z(q_j,0)^{\varepsilon-1}|\mathcal{S}|}\frac{\epsilon}{6}$. Thus $k < \bar{k}$ implies

$$\begin{aligned}
A_2^{k,\Delta}(\mathbf{N}) &= k^{\mathtt{d}}\sum_i\left\{-n_i\left(\underline{R}_i^k - \bar{R}_i^k\right) + z\left(q_j, k^{\mathtt{d}}\sum_i n_i\right)^{\varepsilon-1}\left(\bar{D}_i^k\bar{b}_i^k - \underline{D}_i^k\underline{b}_i^k\right)n_i g\left(\frac{1}{n_i}\right)\right\} \\
&\leq k^{\mathtt{d}}\sum_i\left\{n_i\frac{\epsilon}{6\left(2\bar{N}\right)} + \frac{z\left(q_j,k^{\mathtt{d}}\sum_i n_i\right)^{\varepsilon-1}}{z\left(q_j,0\right)^{\varepsilon-1}|\mathcal{S}|}\frac{\epsilon}{6}n_i g\left(\frac{1}{n_i}\right)\right\} \\
&\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} \\
&= \frac{\epsilon}{3}
\end{aligned}$$

where the third line follows from $k^{\mathtt{d}}\sum_i n_i = \Delta^{\mathtt{d}}\sum_i N_i \leq 2\bar{N}$, $z\left(q_j, k^{\mathtt{d}}\sum_i n_i\right) \leq z\left(q_j,0\right)$, and $n_i g\left(\frac{1}{n_i}\right) \leq 1$. If $\bar{\Delta}_2$ is such that $\Delta < \bar{\Delta}_2$ implies that $K\left(\Delta\right) < \bar{k}$, $\Delta < \bar{\Delta}_2$ implies that $A_2^{K(\Delta),\Delta}\left(\mathbf{N}\right) < \frac{\epsilon}{3}$.

We turn next to the third term. Using $z\left(q_j, k^{\mathtt{d}}\sum_i n_i\right)^{\varepsilon-1}\underline{D}_i^k\underline{b}_i^k \leq z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}$ along with $g\left(\frac{1}{n_i}\right) \geq g\left(\rho\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\frac{1}{n_i}\right)$ (which follows from the fact that g is increasing and $\rho\left(\cdot\right) \leq 1$), we have

$$\begin{aligned}
A_3^{k,\Delta}(\mathbf{N}) &= k^{\mathtt{d}}\sum_i\left\{z\left(q_j, k^{\mathtt{d}}\sum_i n_i\right)^{\varepsilon-1}\underline{D}_i^k\underline{b}_i^k n_i\left[g\left(\frac{1}{n_i}\right) - g\left(\rho\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\frac{1}{n_i}\right)\right]\right\} \\
&\leq z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}k^{\mathtt{d}}\sum_i n_i\left[g\left(\frac{1}{n_i}\right) - g\left(\rho\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\frac{1}{n_i}\right)\right]
\end{aligned}$$

Let $u > 0$ be small enough so that such that $ug\left(\frac{1}{u}\right) < \frac{1}{z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}}\frac{\epsilon}{3}$. Since $ng\left(\frac{1}{n}\right)$ is increasing, we have for any $n_i \leq u$ and any Δ,

$$n_i\left[g\left(\frac{1}{n_i}\right) - g\left(\rho\left(\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\frac{1}{n_i}\right)\right] \leq n_i g\left(\frac{1}{n_i}\right) \leq \frac{1}{z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}}\frac{\epsilon}{3}$$

Let $\bar{\Delta}_3$ be such that $\Delta < \bar{\Delta}_3$ implies $1-\rho\left(\frac{K(\Delta)^{\mathtt{d}}}{\Delta^{\mathtt{d}}}u\right) \leq \frac{1}{z(q_j,0)^{\varepsilon-1}\bar{D}\bar{b}g'(0)}\frac{\epsilon}{3}$. Such a $\bar{\Delta}_3$ exists because $\lim_{x\to\infty}\rho(x) = 1$. For i such that $n_i > u$, we can then bound the term using the fact that g is concave, which implies $g\left(\frac{1}{n}\right) \leq g\left(\rho\left(\cdot\right)\frac{1}{n}\right) + g'\left(\rho\left(\cdot\right)\frac{1}{n}\right)\frac{1}{n_i}\left[1-\rho\left(\cdot\right)\right] \leq g\left(\rho\left(\cdot\right)\frac{1}{n}\right) + g'(0)\frac{1}{n_i}\left[1-\rho\left(\cdot\right)\right]$. Together, these imply for $n_i > u$ and $\Delta < \bar{\Delta}_3$

$$\begin{aligned} n_i\left[g\left(\frac{1}{n_i}\right) - g\left(\rho\left(\frac{K\left(\Delta\right)^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\frac{1}{n_i}\right)\right] &\leq g'(0)\left[1-\rho\left(\frac{K\left(\Delta\right)^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\right] \\ &\leq g'(0)\left[1-\rho\left(\frac{K\left(\Delta\right)^{\mathtt{d}}}{\Delta^{\mathtt{d}}}u\right)\right] \\ &\leq g'(0)\frac{1}{z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}g'(0)}\frac{\epsilon}{3} \end{aligned}$$

where the second inequality used the fact that ρ is increasing and $n_i \geq u$. Together, these imply that for any n_i and $\Delta < \bar{\Delta}_3$,

$$n_i\left[g\left(\frac{1}{n_i}\right) - g\left(\rho\left(\frac{K\left(\Delta\right)^{\mathtt{d}}}{\Delta^{\mathtt{d}}}n_i\right)\frac{1}{n_i}\right)\right] \leq \frac{1}{z\left(q_j,0\right)^{\varepsilon-1}\bar{D}\bar{b}}\frac{\epsilon}{3}$$

As a result, if $\Delta < \bar{\Delta}_3$, then $A_3^{K(\Delta),\Delta}\left(\mathbf{N}\right) \leq \frac{\epsilon}{3}$.

These three results together imply that if $\Delta < \min\left\{\bar{\Delta}_1, \bar{\Delta}_2, \bar{\Delta}_3\right\}$, then

$$\left|\bar{U}^{k,\Delta}\left(\mathbf{N}\right) - \underline{U}^{k,\Delta}\left(\mathbf{N}\right)\right| < \epsilon$$

■

Lemma A.25 *For any $m > 0$, there is an $\bar{\Delta} > 0$ such that $\Delta < \bar{\Delta}$ implies $\Delta^{\mathtt{d}}\left|O^{\Delta*}\right| \leq \bar{N} + m$ for any optimal choice $O^{\Delta*}$.*

Proof. Fix $m > 0$. For any k, Δ, consider a vector $\mathbf{N} = \{N_i\}_{i\in I^k}$, along with the alternative vector $\tilde{\mathbf{N}} = \left\{\tilde{N}_i\right\}_{i\in I^k}$ where $\tilde{N}_i = \min\left\{N_i, \left\lceil\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}}\bar{n}\right\rceil\right\}$. We first derive an upper bound on $\bar{U}^{k,\Delta}\left(\mathbf{N}\right) - \bar{U}^{k,\Delta}\left(\tilde{\mathbf{N}}\right)$. Define $n_i = \frac{\Delta^{\mathtt{d}}}{k^{\mathtt{d}}}N_i$ and $\tilde{n}_i = \frac{\Delta^{\mathtt{d}}}{k^{\mathtt{d}}}\tilde{N}_i$. Define $a^{k,\Delta} \equiv \frac{\Delta^{\mathtt{d}}}{\bar{n}k^{\mathtt{d}}}\left\lceil\frac{k^{\mathtt{d}}}{\Delta^{\mathtt{d}}}\bar{n}\right\rceil$, so that $\tilde{n}_i = \min\left\{n_i, a^{k,\Delta}\bar{n}\right\}$. There is a $\bar{\Delta}_1$ such that $\Delta < \bar{\Delta}_1$ implies $a^{K(\Delta),\Delta} < 1 + \frac{m}{3\bar{N}}$.

Noting that $z\left(q_j, k^{\mathtt{d}} \sum_i \tilde{n}_i\right) \geq z\left(q_j, k^{\mathtt{d}} \sum_i n_i\right)$, (and using the shorthand $\tilde{z} \equiv z\left(q_j, k^{\mathtt{d}} \sum_i \tilde{n}_i\right)$), we have:

$$\begin{aligned}
\bar{U}^{k,\Delta}(\mathbf{N}) - \bar{U}^{k,\Delta}\left(\tilde{\mathbf{N}}\right) &= k^{\mathtt{d}} \sum_{i \in I^k} \left\{ -n_i \underline{R}_i^k + z\left(q_j, k^{\mathtt{d}} \sum_i n_i\right)^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \kappa(n_i) \right\} \\
&\quad + z(q_j, 0)^{\varepsilon-1} \bar{D}\bar{b} \frac{1}{k} \int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta - \bar{U}^{k,\Delta}\left(\tilde{\mathbf{N}}\right) \\
&\leq k^{\mathtt{d}} \sum_{i \in I^k} \left\{ -n_i \underline{R}_i^k + \tilde{z}^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \kappa(n_i) \right\} + z(q_j, 0)^{\varepsilon-1} \bar{D}\bar{b} \frac{1}{k} \int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta - \bar{U}^{k,\Delta}\left(\tilde{\mathbf{N}}\right) \\
&= k^{\mathtt{d}} \sum_{i \in I^k} \left\{ -n_i \underline{R}_i^k + \tilde{z}^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \kappa(n_i) \right\} - k^{\mathtt{d}} \sum_{i \in I^{k+}} \left\{ -\tilde{n}_i \underline{R}_i^k + \tilde{z}^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \kappa(\tilde{n}_i) \right\} \\
&= k^{\mathtt{d}} \sum_{i \in I^{k+}} \left\{ -\left(n_i - a^{k,\Delta}\bar{n}\right) \underline{R}_i^k + \tilde{z}^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \left[\kappa(n_i) - \kappa\left(a^{k,\Delta}\bar{n}\right)\right] \right\}
\end{aligned}$$

where I^{k+} is the set of squares such that $N_i > \tilde{N}_i$ (and hence $n_i > \tilde{n}_i = a^{k,\Delta}\bar{n}$). The concavity of κ and $a^{k,\Delta} \geq 1$ implies that $\kappa(n_i) - \kappa\left(a^{k,\Delta}\bar{n}\right) \leq \kappa'\left(a^{k,\Delta}\bar{n}\right)\left(n_i - a^{k,\Delta}\bar{n}\right) \leq \kappa'(\bar{n})\left(n_i - a^{k,\Delta}\bar{n}\right) = \frac{1}{2}\frac{\underline{R}}{z(q,0)^{\varepsilon-1}\bar{D}\bar{b}}\left(n_i - a^{k,\Delta}\bar{n}\right)$. This gives

$$\begin{aligned}
\bar{U}^{k,\Delta}(\mathbf{N}) - \bar{U}^{k,\Delta}\left(\tilde{\mathbf{N}}\right) &\leq k^{\mathtt{d}} \sum_{i \in I^{k+}} \left\{ -\left(n_i - a^{k,\Delta}\bar{n}\right) \underline{R}_i^k + \frac{1}{2} \tilde{z}^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \frac{\underline{R}}{z(q,0)^{\varepsilon-1} \bar{D}\bar{b}} \left(n_i - a^{k,\Delta}\bar{n}\right) \right\} \\
&\leq k^{\mathtt{d}} \sum_{i \in I^{k+}} \left[-1 + \frac{1}{2}\right] \underline{R} \left(n_i - a^{k,\Delta}\bar{n}\right) \\
&= -\frac{\underline{R}}{2} k^{\mathtt{d}} \sum_{i \in I^{k+}} \left(n_i - a^{k,\Delta}\bar{n}\right)
\end{aligned}$$

In particular, if $\Delta^{\mathtt{d}} \sum_{i \in I^k} N_i \geq \bar{N} + m$ and $\Delta < \bar{\Delta}_1$, then

$$\begin{aligned}
K(\Delta)^{\mathtt{d}} \sum_{i \in I^{K(\Delta)+}} \left(n_i - a^{K(\Delta),\Delta}\bar{n}\right) &\geq K(\Delta)^{\mathtt{d}} \sum_{i \in I^{K(\Delta)}} \left(n_i - a^{K(\Delta),\Delta}\bar{n}\right) \\
&= K(\Delta)^{\mathtt{d}} \sum_{i \in I^{K(\Delta)}} n_i - a^{K(\Delta),\Delta}\bar{N} \\
&= \Delta^{\mathtt{d}} \sum_{i \in I^{K(\Delta)}} N_i - a^{K(\Delta),\Delta}\bar{N} \\
&\geq \bar{N} + m - a^{K(\Delta),\Delta}\bar{N} \\
&\geq \bar{N} + m - \left(1 + \frac{m}{3\bar{N}}\right)\bar{N} \\
&\geq \frac{2}{3}m
\end{aligned}$$

which would imply

$$\bar{U}^{K(\Delta),\Delta}(\mathbf{N}) - \bar{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}\right) \leq -\frac{R}{3}m$$

Let $\bar{\Delta}_2 > 0$ be such that for any $\Delta < \bar{\Delta}_2$ and any $\tilde{\mathbf{N}} = \left\{\tilde{N}_i\right\}_{i \in I^{K(\Delta)}}$ such that $\sum_{i \in I^{K(\Delta)}} \tilde{N}_i \leq \frac{2\bar{N}}{\Delta^{\mathsf{d}}}$,

$$\left|\bar{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}\right) - \underline{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}\right)\right| < \frac{R}{4}m$$

. The existence of Such a Δ follows from Lemma A.24.

For any $\Delta < \bar{\Delta} = \min\{\bar{\Delta}_1, \Delta_2\}$, let $O^{\Delta *}$ be among the optimal solutions. Define $\mathbf{N}^{\Delta} = \left\{N_i^{\Delta}\right\}_{i \in I^{K(\Delta)}}$ to be such that $N_i^{\Delta} = \left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|$. Similarly, define $\tilde{\mathbf{N}}^{\Delta} = \left\{\tilde{N}_i^{\Delta}\right\}_{i \in I^{K(\Delta)}}$ where $\tilde{N}_i^{\Delta} = \min\left\{N_i^{\Delta}, \left\lceil \frac{K(\Delta)^{\mathsf{d}}}{\Delta^{\mathsf{d}}}\bar{n} \right\rceil\right\}$.

Toward a contradiction, suppose that $\Delta^{\mathsf{d}}\left|O^{\Delta *}\right| \geq \bar{N} + m$. The first step of the proof implies

$$\bar{U}^{K(\Delta),\Delta}\left(\mathbf{N}^{\Delta}\right) - \bar{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) \leq -\frac{R}{3}m$$

Finally, we have that $\Pi^{\Delta}\left(O^{\Delta *}\right) = U^{K(\Delta),\Delta}\left(\mathbf{N}^{\Delta}\right) \leq \bar{U}^{K(\Delta),\Delta}\left(\mathbf{N}^{\Delta}\right)$, and $U^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) \geq \underline{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right)$. Together, these imply that if $\Delta < \bar{\Delta}$,

$$\begin{aligned} U^{K(\Delta),\Delta}\left(\mathbf{N}^{\Delta}\right) - U^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) &\leq \bar{U}^{K(\Delta),\Delta}\left(\mathbf{N}^{\Delta}\right) - \underline{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) \\ &= \bar{U}^{K(\Delta),\Delta}\left(\mathbf{N}^{\Delta}\right) - \bar{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) + \bar{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) - \underline{U}^{K(\Delta),\Delta}\left(\tilde{\mathbf{N}}^{\Delta}\right) \\ &\leq -\frac{R}{3}m + \frac{R}{4}m \\ &< 0 \end{aligned}$$

Therefore, $O^{\Delta *}$ cannot be optimal, a contradiction. ■

Lemma A.26 *Fix $\epsilon > 0$. There exists a $\bar{\Delta}$ such that for any $\Delta < \bar{\Delta}$ and any $\mathbf{N} = \{N_i\}_{i \in I^{K(\Delta)}}$ such that $\sum_{i \in I^{K(\Delta)}} N_i \leq 2\bar{N}$,*

$$\left|U^{K(\Delta),\Delta}(\mathbf{N}) - \Pi\left(\hat{n}^{K(\Delta),\Delta}(\mathbf{N})\right)\right| < \epsilon$$

Proof. The first step in this proof is to show that for any k, Δ, and any admissible $\mathbf{N} = \{N_i\}_{i \in I^k}$, we can bound $\left|U^{k,\Delta}(\mathbf{N}) - \Pi\left(\hat{n}^{k,\Delta}(\mathbf{N})\right)\right|$. We already know that

$$\underline{U}^{k,\Delta}(\mathbf{N}) \leq U^{k,\Delta}(\mathbf{N}) \leq \bar{U}^{k,\Delta}(\mathbf{N})$$

In addition, we have

$$\underline{U}^{k,\Delta}(\mathbf{N}) \leq \Pi\left(\hat{n}^{k,\Delta}(\mathbf{N})\right) \leq \bar{U}^{k,\Delta}(\mathbf{N})$$

because

$$
\begin{aligned}
\Pi\left(\hat{n}^{k,\Delta}(\mathbf{N})\right) &= \max_n \int_{s\in\mathcal{S}} \left\{-R_s n_s + D_s b_s z \left(q, \sum_{i\in I^k} N_i\right)^{\varepsilon-1} \kappa(n_s)\right\} ds \;\; \text{subject to} \int_{s\in\mathcal{S}_i^k} n_s ds = N_i \\
&\leq \max_n \int_{s\in\mathcal{S}} \left\{-\underline{R}_i^k n_s + \bar{D}_i^k \bar{b}_i^k z \left(q, \sum_{i\in I^k} N_i\right)^{\varepsilon-1} \kappa(n_s)\right\} ds \;\; \text{subject to} \int_{s\in\mathcal{S}_i^k} n_s ds = N_i \\
&\leq \bar{U}^{k,\Delta}(\mathbf{N})
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi\left(\hat{n}^{k,\Delta}(\mathbf{N})\right) &= \max_n \int_{s\in\mathcal{S}} \left\{-R_s n_s + D_s b_s z \left(q, \sum_{i\in I^k} N_i\right)^{\varepsilon-1} \kappa(n_s)\right\} ds \;\; \text{subject to} \int_{s\in\mathcal{S}_i^k} n_s ds = N_i \\
&\geq \max_n \int_{s\in\mathcal{S}} \left\{-\bar{R}_i^k n_s + \underline{D}_i^k \underline{b}_i^k z \left(q, \sum_{i\in I^k} N_i\right)^{\varepsilon-1} \kappa(n_s)\right\} ds \;\; \text{subject to} \int_{s\in\mathcal{S}_i^k} n_s ds = N_i \\
&\geq \underline{U}^{k,\Delta}(\mathbf{N})
\end{aligned}
$$

Together, these imply that

$$
\left|U^{K(\Delta),\Delta}(\mathbf{N}) - \Pi\left(\hat{n}^{K(\Delta),\Delta}(\mathbf{N})\right)\right| \leq \left|\bar{U}^{K(\Delta),\Delta}(\mathbf{N}) - \underline{U}^{K(\Delta),\Delta}(\mathbf{N})\right|
$$

Finally, Lemma A.24 gives that there is a $\bar{\Delta}$ such that $\left|\bar{U}^{K(\Delta),\Delta}(\mathbf{N}) - \underline{U}^{K(\Delta),\Delta}(\mathbf{N})\right| \leq \epsilon$, and hence

$$
\left|U^{K(\Delta),\Delta}(\mathbf{N}) - \Pi\left(\hat{n}^{K(\Delta),\Delta}(\mathbf{N})\right)\right| < \epsilon
$$

■

Lemma A.27 *For any $\epsilon > 0$, there is a $\bar{\Delta}$ small enough so that $\Delta < \bar{\Delta}$ implies*

$$
\sum_{i\in I^{K(\Delta)}} \left|\Delta^{\mathtt{d}} \left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right| - \int_{s\in\mathcal{S}_i^{K(\Delta)}} n_s^* ds\right| < \epsilon
$$

Proof. For any Δ, define $n^{\Delta *} \equiv \arg\max_n \Pi(n)$ subject to $\int_{s\in\mathcal{S}_i^{K(\Delta)}} n_s ds = \Delta^{\mathtt{d}} \left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|$.

We first show that, for any $\eta > 0$, there is an $\bar{\Delta}$ small enough so that $\Delta < \bar{\Delta}$ implies that $\left|\Pi(n^*) - \Pi\left(n^{\Delta *}\right)\right| \leq$

η. For any Δ, optimality implies both

$$
\begin{aligned}
0 &\leq \Pi\left(n^*\right) - \Pi\left(n^{\Delta *}\right) \\
0 &\leq U^{K(\Delta),\Delta}\left(\left\{\left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|\right\}_{i \in I^{K(\Delta)}}\right) - U^{K(\Delta),\Delta}\left(\left\{\frac{1}{\Delta^{\mathsf{d}}}\int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds\right\}_{i \in I^{K(\Delta)}}\right)
\end{aligned}
$$

Adding these together and rearranging gives

$$
\begin{aligned}
0 &\leq \Pi\left(n^*\right) - \Pi\left(n^{\Delta *}\right) + U^{K(\Delta),\Delta}\left(\left\{\left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|\right\}_{i \in I^{K(\Delta)}}\right) - U^{K(\Delta),\Delta}\left(\left\{\frac{1}{\Delta^{\mathsf{d}}}\int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds\right\}_{i \in I^{K(\Delta)}}\right) \\
&= \Pi\left(n^*\right) - U^{K(\Delta),\Delta}\left(\left\{\frac{1}{\Delta^{\mathsf{d}}}\int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds\right\}_{i \in I^{K(\Delta)}}\right) - \Pi\left(n^{\Delta *}\right) + U^{K(\Delta),\Delta}\left(\left\{\left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|\right\}_{i \in I^{K(\Delta)}}\right) \\
&\leq \left|\Pi\left(n^*\right) - U^{K(\Delta),\Delta}\left(\left\{\frac{1}{\Delta^{\mathsf{d}}}\int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds\right\}_{i \in I^{K(\Delta)}}\right)\right| + \left|\Pi\left(n^{\Delta *}\right) - U^{K(\Delta),\Delta}\left(\left\{\left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|\right\}_{i \in I^{K(\Delta)}}\right)\right|
\end{aligned}
$$

Note that $\int_{s\in\mathcal{S}} n_s^* ds \leq \bar{N}$ and, for sufficiently small Δ, $\Delta^{\mathsf{d}}\left|O^{\Delta *}\right| \leq 2\bar{N}$ (the latter uses Lemma A.25 with $m = \bar{N}$). Lemma A.26 therefore implies that there is a $\bar{\Delta}$ such that $\Delta < \bar{\Delta}$ implies that each term is less than $\frac{\eta}{2}$. As a result, $0 \leq \Pi\left(n^*\right) - \Pi\left(n^{\Delta *}\right) \leq \eta$.

Claim A.23 states that for any $\epsilon > 0$, there is an η such that $\left|\Pi\left(n^*\right) - \Pi\left(n^{\Delta *}\right)\right| < \eta$ implies $\left\|n^* - n^{\Delta *}\right\| < \epsilon$. Thus there is an $\bar{\Delta}$ small enough so that $\Delta < \bar{\Delta}$ implies that $\left|\Pi\left(n^*\right) - \Pi\left(n^{\Delta *}\right)\right| < \eta$ and hence $\left\|n^* - n^{\Delta *}\right\| < \epsilon$. Finally, the definition of $n^{\Delta *}$ implies $\int_{s\in\mathcal{S}_i^{K(\Delta)}} n_s^{\Delta *} ds = \Delta^{\mathsf{d}}\left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right|$. Thus if $\Delta < \bar{\Delta}$, so we have

$$
\begin{aligned}
\sum_{i \in I^{K(\Delta)}}\left|\Delta^{\mathsf{d}}\left|O^{\Delta *} \cap \mathcal{S}_i^{K(\Delta)}\right| - \int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds\right| &= \sum_{i \in I^{K(\Delta)}}\left|\int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^{\Delta *} ds - \int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds\right| \\
&\leq \sum_{i \in I^{K(\Delta)}} \int_{s \in \mathcal{S}_i^{K(\Delta)}} \left|n_s^{\Delta *} - n_s^*\right| ds \\
&= \left\|n^{\Delta *} - n^*\right\| \\
&< \epsilon
\end{aligned}
$$

■

Proposition A.28 *Consider any Jordan measurable set $\mathcal{A}$. For any ϵ, there is a $\bar{\Delta}$ such that $\Delta < \bar{\Delta}$ implies that $\left|\Delta^{\mathsf{d}}\left|O^{\Delta *} \cap \mathcal{A}\right| - \int_{s\in\mathcal{A}} n_s^* ds\right| < \epsilon$.*

Proof. Let $\bar{I}^k(\mathcal{A}) = \left\{i \in I^k \text{ such that } \mathcal{S}_i^k \cap \mathcal{A} \neq \emptyset\right\}$ be the smallest collection of squares/segments that contains $\mathcal{A}$, and let $\underline{I}^k(\mathcal{A}) = \left\{i \in I^k \text{ such that } \mathcal{S}_i^k \subseteq \mathcal{A}\right\}$ be the largest collection contained in A. Let $\mathcal{B}^k(\mathcal{A}) = \cup_{i\in \bar{I}^k(\mathcal{A}) \setminus \underline{I}^k(\mathcal{A})} \mathcal{S}_i^k$ be the union of segments/squares that contains points in both $\mathcal{A}$ and its complement.

Consider and optimal solution $O^{\Delta *}$ for economy Δ. For any k, define $N_i^{k\Delta} \equiv \left|O^{\Delta *} \cap S_i^k\right|$ to be the number of plants in segment/square S_i^k. We have

$$\sum_{i \in \underline{I}^k(\mathcal{A})} N_i^{k\Delta} \leq \left|O^{\Delta *} \cap \mathcal{A}\right| \leq \sum_{i \in \bar{I}^k(\mathcal{A})} N_i^{k\Delta}$$

By multiplying through by $\Delta^{\mathtt{d}}$ and subtracting $\int_{s \in \mathcal{A}} n_s^* ds$ from each side, the left hand inequality can be expressed as

$$\begin{aligned}
\Delta^{\mathtt{d}} \left|O^{\Delta *} \cap \mathcal{A}\right| - \int_{s \in \mathcal{A}} n_s^* ds &\geq \sum_{i \in \underline{I}^k(\mathcal{A})} \Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{A}} n_s^* ds \\
&\geq \sum_{i \in \underline{I}^k(\mathcal{A})} \left(\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right) - \int_{s \in \mathcal{B}^k(\mathcal{A})} n_s^* ds \\
&\geq -\sum_{i \in \underline{I}^k(\mathcal{A})} \left|\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right| - \int_{s \in \mathcal{B}^k(\mathcal{A})} n_s^* ds \\
&\geq -\sum_{i \in \bar{I}^k(\mathcal{A})} \left|\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right| - \int_{s \in \mathcal{B}^k(\mathcal{A})} n_s^* ds
\end{aligned}$$

Similarly, by multiplying through by $\Delta^{\mathtt{d}}$ and subtracting $\int_{s \in \mathcal{A}} n_s^* ds$ from each side, the right hand inequality can be expressed as

$$\begin{aligned}
\Delta^{\mathtt{d}} \left|O^{\Delta *} \cap \mathcal{A}\right| - \int_{s \in \mathcal{A}} n_s^* ds &\leq \sum_{i \in \bar{I}^k(\mathcal{A})} \Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{A}} n_s^* ds \\
&\leq \sum_{i \in \bar{I}^k(\mathcal{A})} \left(\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right) + \int_{s \in \mathcal{B}^k(\mathcal{A})} n_s^* ds \\
&\leq \sum_{i \in \bar{I}^k(\mathcal{A})} \left|\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right| + \int_{s \in \mathcal{B}^k(\mathcal{A})} n_s^* ds
\end{aligned}$$

Together, these give

$$\begin{aligned}
\left|\Delta^{\mathtt{d}} \left|O^{\Delta *} \cap \mathcal{A}\right| - \int_{s \in \mathcal{A}} n_s^* ds\right| &\leq \sum_{i \in \bar{I}^k(\mathcal{A})} \left|\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right| + \int_{s \in \mathcal{B}^k(\mathcal{A})} n_s^* ds \\
&\leq \sum_{i \in I^k} \left|\Delta^{\mathtt{d}} N_i^{k\Delta} - \int_{s \in \mathcal{S}_i^k} n_s^* ds\right| + \bar{n} \left|\mathcal{B}^k(\mathcal{A})\right|
\end{aligned}$$

In particular this bound holds for any Δ, $K(\Delta)$ pair.

From Lemma A.27, there is a $\bar{\Delta}_1$ small enough so that $\Delta < \bar{\Delta}_1$ implies that

$$\sum_{i \in I^{K(\Delta)}} \left| \Delta^{\mathsf{d}} N_i^{K(\Delta)\Delta} - \int_{s \in \mathcal{S}_i^{K(\Delta)}} n_s^* ds \right| < \frac{\epsilon}{2}.$$

Further, since $\mathcal{A}$ is Jordan measurable, $\lim_{k \to 0} \mathcal{B}^k(\mathcal{A}) = 0$, so there is a $\bar{\Delta}_2$ small enough so that $\Delta < \bar{\Delta}_2$ implies $\left|\mathcal{B}^{K(\Delta)}(\mathcal{A})\right| < \frac{1}{\bar{n}} \frac{\epsilon}{2}$. Together, these imply that $\Delta < \min\left\{\bar{\Delta}_1, \bar{\Delta}_2\right\}$ implies that

$$\left| \Delta^{\mathsf{d}} \left| O^{\Delta *} \cap \mathcal{A} \right| - \int_{s \in \mathcal{A}} n_s^* ds \right| < \epsilon.$$

■

A.4 Additional Proofs

We formally state and prove here the additional result, quoted in the text, about the relative marginal efficiency of distribution.

Lemma A.29 *Consider two firms with $z_1 < z_2$ and two locations with $R_s < R_{\hat{s}}$. Then, if n_{1s}, $n_{1\hat{s}}$, n_{2s}, $n_{2\hat{s}} > 0$,*

$$\frac{\kappa'(n_{2s})}{\kappa'(n_{1s})} > \frac{\kappa'(n_{2\hat{s}})}{\kappa'(n_{1\hat{s}})}.$$

Proof. Since $\lambda_2 > \lambda_1$, $\frac{R+\lambda_2}{R+\lambda_1} > 1$, so $\frac{R+\lambda_2}{R+\lambda_1}$ is decreasing in R. Hence, we have that

$$\frac{\kappa'(n_{2s})}{\kappa'(n_{1s})} = \frac{z_1^{\varepsilon-1}}{z_2^{\varepsilon-1}} \frac{R_s + \lambda_2}{R_s + \lambda_1} > \frac{z_1^{\varepsilon-1}}{z_2^{\varepsilon-1}} \frac{R_{\hat{s}} + \lambda_2}{R_{\hat{s}} + \lambda_1} = \frac{\kappa'(n_{2\hat{s}})}{\kappa'(n_{1\hat{s}})}.$$

■

A.4.1 Proof of Proposition 6

Proof. $n_s(z)$ denotes the density of plants a firm with productivity z places in location s. The first order condition (4) implies that $n_s(z) = 0$ if $\frac{R_s+\lambda_j}{x_s z^{\varepsilon-1}} > \kappa'(0)$. Since $\lambda_j > 0$,

$$\lim_{z \to 0} \frac{R + \lambda_j}{x z^{\varepsilon-1}} \geq \lim_{z \to 0} \frac{R}{x z^{\varepsilon-1}} = \infty,$$

and, if $\lim_{z \to \infty} \frac{\lambda_j}{z^{\varepsilon-1}} = \infty$,

$$\lim_{z \to \infty} \frac{R + \lambda_j}{x z^{\varepsilon-1}} = \frac{1}{x} \lim_{z \to \infty} \frac{\lambda_j}{z^{\varepsilon-1}} = \infty.$$

The result follows from the fact that κ' is continuous, strictly decreasing, and $\kappa'(0) < \infty$ if $\lim_{\delta \to \infty} \frac{\delta^{\mathsf{d}}}{t(\delta)^{\varepsilon-1}} = 0$ by Lemma 3. ■

A.4.2 Proof of Lemma 7

Proof. For firm j, variable profit in location s is $x_s z_j^{\varepsilon-1}\kappa(n_{js})$, so with a markup of $\frac{\varepsilon}{\varepsilon-1}$, the expenditure on labor in s is $W_s l_{js} = (\varepsilon-1)x_s z_j^{\varepsilon-1}\kappa(n_{js})$. Since the wage $W_s = W$ for all $s \in \mathcal{S}$, j's total employment is $L_j = \frac{\varepsilon-1}{W}\int x_s z_j^{\varepsilon-1}\kappa(n_{js})\,ds = \frac{\sigma}{W}\lambda_j$. If $N_1 = N_2 = 0$, then $L_1 = L_2 = 0$. Otherwise, by Lemma 4, $\frac{\lambda_2}{z_2^{\varepsilon-1}} > \frac{\lambda_1}{z_1^{\varepsilon-1}}$, which implies $\lambda_2 > \lambda_1$, and so $L_2 > L_1$. ■

A.5 Proof of Proposition 8: Aggregation

In this appendix we prove proposition Proposition 8 and show some additional aggregate properties of the industry equilibrium defined in Section 3.

A.5.1 The Local Price Index

The price that firm j sets in location s is

$$p_{js} = \frac{\varepsilon}{\varepsilon-1}\min_{o\in O_j}\left\{\frac{W_o T(\delta_{so})}{B_o Z(q, N_j)}\right\}$$

In a small enough neighborhood of location s, economic activity is locally uniform. Thus each firm will choose to have catchment areas that are locally uniform regular hexagons. Among firms with the same effective productivity Z, the pattern of plant locations will be the same up to translation. These translations are such that if we integrate across such firms, the total measure of plants at each point will be uniform. An implications is that, for consumers in location s and firms with effective productivity Z, the fraction of those firms that have plants closer than distance δ to those consumers is the same as the fraction of locations in such a firm's catchment area that are closer than distance δ to the plant at the center of the catchment area.

Given this we now derive an expression for the ideal price index at a location. As in the proof of the main proposition, we will proceed by dividing the economy into $k \times k$ squares in which economic activity is uniform, taking the limit as $\Delta \to 0$, and then taking the limit as $k \to 0$. We ignore boundary issues because these will disappear when we take the limit as $\Delta \to 0$.

The ideal price index at location s satisfies $P_s^{1-\varepsilon} = \int p_{js}^{1-\varepsilon}dj$. Consider a $k \times k$ square with uniform local economic activity, so that catchment areas are uniform hexagons. We can compute the local ideal price index at any point in that $k \times k$ square by integrating over all firms in the economy.

Let N_{ji} be the number of plants that firm j places in the square. Then for each plant, the distance to the furthest point in the catchment area is $\psi\sqrt{k^2/N_{ji}}$, and among points that are distance δ from the plant, the fraction $\varpi\left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}}\right)$ are in the plant's catchment area (the remainder are served by other plants).[43]

[43] Recall that $\varpi(x)$ is defined as that fraction of a circle of radius x that intersects with the interior of a hexagon with side length 1.

The ideal price index can therefore be expressed as

$$\left(P_s^{k\Delta}\right)^{1-\varepsilon} = \int \frac{\int_0^{\psi\sqrt{k^2/N_{ji}}} \varpi\left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}}\right)\left[\frac{\varepsilon}{\varepsilon-1}\frac{W_s T(\delta)}{B_s Z(q,N_j)}\right]^{1-\varepsilon} 2\pi\delta d\delta}{\int_0^{\psi\sqrt{k^2/N_{ji}}} \varpi\left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}}\right) 2\pi\delta d\delta} dj$$

Using $T(\delta) = t\left(\frac{\delta}{\Delta}\right)$ and $Z\left(q,N\right) \equiv z\left(q,\Delta^{\mathtt{d}}N\right)^{\varepsilon-1}$ gives

$$\left(P_s^{k\Delta}\right)^{1-\varepsilon} = \int \frac{\int_0^{\psi\sqrt{k^2/N_{ji}}} \varpi\left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}}\right)\left[\frac{\frac{\varepsilon}{\varepsilon-1}W_s t\left(\frac{\delta}{\Delta}\right)}{B_s z(q,\Delta^{\mathtt{d}}N_j)}\right]^{1-\varepsilon} 2\pi\delta d\delta}{\int_0^{\psi\sqrt{k^2/N_{ji}}} \varpi\left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}}\right) 2\pi\delta d\delta} dj$$

Using $n_{ji} = \frac{\Delta^2 N_{ji}}{k^2}$ and using a change of variables gives

$$\left(P_s^{k\Delta}\right)^{1-\varepsilon} = \left(\frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s}\right)^{1-\varepsilon}\int z_j^{\varepsilon-1}\frac{\int_0^{\psi n_{ji}^{-1/2}} \varpi\left(\frac{\tilde{\delta}}{\psi n_{ji}^{-1/2}}\right) t\left(\tilde{\delta}\right)^{1-\varepsilon} 2\pi\tilde{\delta}d\tilde{\delta}}{\int_0^{\psi n_{ji}^{-1/2}} \varpi\left(\frac{\tilde{\delta}}{\psi n_{ji}^{-1/2}}\right) 2\pi\tilde{\delta}d\tilde{\delta}} dj$$

Taking the limits as $\Delta \to 0$ and $k \to 0$ gives

$$\begin{aligned}
P_s^{1-\varepsilon} &= \left(\frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s}\right)^{1-\varepsilon}\int z_j^{\varepsilon-1}\frac{\int_0^{\psi n_{js}^{-1/2}} \varpi\left(\frac{\tilde{\delta}}{\psi n_{js}^{-1/2}}\right) t\left(\tilde{\delta}\right)^{1-\varepsilon} 2\pi\tilde{\delta}d\tilde{\delta}}{\int_0^{\psi n_{js}^{-1/2}} \varpi\left(\frac{\tilde{\delta}}{\psi n_{js}^{-1/2}}\right) 2\pi\tilde{\delta}d\tilde{\delta}} dj \\
&= \left(\frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s}\right)^{1-\varepsilon}\int z_j^{\varepsilon-1} n \int_0^{\psi n_{js}^{-1/2}} \varpi\left(\frac{\tilde{\delta}}{\psi n_{js}^{-1/2}}\right) t\left(\tilde{\delta}\right)^{1-\varepsilon} 2\pi\tilde{\delta}d\tilde{\delta}dj \\
&= \left(\frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s}\right)^{1-\varepsilon}\int z_j^{\varepsilon-1}\kappa\left(n_{js}\right)dj
\end{aligned}$$

Define $\mathcal{Z}_s^{\varepsilon-1} \equiv \left(\int z_j^{\varepsilon-1}\kappa\left(n_{js}\right)dj\right)$

$$P_s = \frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s\mathcal{Z}_s}$$

With this, we can simplify the expression for local profitability, x_s:

$$\begin{aligned} x_s &= \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\mathcal{L}_s c_s P_s^{\varepsilon}\left(B_s/W_s\right)^{\varepsilon-1} = \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\mathcal{L}_s P_s c_s \frac{\left(B_s/W_s\right)^{\varepsilon-1}}{P_s^{1-\varepsilon}} = \frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\mathcal{L}_s P_s c_s \frac{\left(B_s/W_s\right)^{\varepsilon-1}}{\left(\frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s\mathcal{Z}_s}\right)^{1-\varepsilon}} \\ &= \frac{1}{\varepsilon}\mathcal{L}_s P_s c_s \left(\mathcal{Z}_s\right)^{1-\varepsilon} \\ &= \frac{1}{\varepsilon}\mathcal{L}_s \frac{\varepsilon}{\varepsilon-1}\frac{W_s}{B_s\mathcal{Z}_s} c_s \left(\mathcal{Z}_s\right)^{1-\varepsilon} \\ &= \frac{1}{\varepsilon-1}\frac{W_s\mathcal{L}_s}{B_s\mathcal{Z}_s^{\varepsilon}} c_s \end{aligned}$$

A.5.2 Market clearing for Space

We use the same approach to characterize the total amount of local real estate used by plants. Consider a square of size $k \times k$. The fraction of of land devoted to commercial real estate is

$$\mathcal{N}_s^{k\Delta} = \xi \frac{1}{k^2}\int_{s\in S_i^k}\int 1\left\{j \text{ has plant in } s\right\} dj ds = \xi\frac{1}{k^2}\int N_{ji}\left(j\right) dj$$

where N_{ji} is the number of plants the firm places in square $\mathcal{S}_i^k$. Using $\xi = \Delta^2$, this is

$$\mathcal{N}_s^{k\Delta} = \Delta^{\mathsf{d}}\frac{1}{k^2}\int N_i^k\left(j\right) dj$$

Using $n_{ji} = \frac{\Delta^2 N_{ji}}{k^2}$

$$\mathcal{N}_s^{k\Delta} = \int n_{ji} dj$$

Taking the limit as $\Delta \to 0$ and $k \to 0$ gives

$$\mathcal{N}_s = \int n_{js} dj$$

A.5.3 Consumption

We derive here an expression for the local consumption bundle. Labor used by firm j in a plant located in o to produce $c_{js}\mathcal{L}_s$ units of output for consumption by households in location s is

$$\begin{aligned} l_{jos}(\delta) &= \frac{T(\delta_{os})}{B_o Z_j} c_{js}\mathcal{L}_s = \frac{T(\delta_{os})}{B_o Z_j} c_s P_s^{\varepsilon} p_{js}^{-\varepsilon}\mathcal{L}_s = \frac{T(\delta_{os})}{B_s Z_j} c_s P_s^{\varepsilon}\left[\frac{\varepsilon}{\varepsilon-1}\frac{W_o T(\delta_{os})}{B_s Z_j}\right]^{-\varepsilon}\mathcal{L}_s \\ &= \left(\frac{\varepsilon}{\varepsilon-1}\right)^{-\varepsilon}\frac{1}{W_o}\left(\frac{W_o T(\delta_{os})}{B_o Z_j}\right)^{1-\varepsilon} c_s P_s^{\varepsilon}\mathcal{L}_s \end{aligned}$$

We again use the approach of studying a $k \times k$ square in which economic activity is uniform. In such a square, firm j sets up N_{ji} plants, each with a catchment area that is a regular hexagon of size $1/N_{ji}$ (again, ignoring boundary issues, which disappear in the limit as $\Delta \to 0$. If the density of employment in the square is $\mathcal{L}_i^{k\Delta}$ and consumption per capita is $c_i^{k\Delta}$, then, per unit of space, total employment of firm in the square is then

$$\frac{1}{k^2} N_{ji} \int_0^{\psi\sqrt{k^2/N_{ji}}} \left[\left(\frac{\varepsilon}{\varepsilon-1} \right)^{-\varepsilon} \frac{1}{W_i^{k\Delta}} \left(\frac{W_i^{k\Delta} T(\delta)}{B_i^{k\Delta} Z_j} \right)^{1-\varepsilon} c_i^{k\Delta} \left(P_i^{k\Delta} \right)^{\varepsilon} \mathcal{L}_i^{k\Delta} \right] \varpi \left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}} \right) 2\pi\delta d\delta$$

Employment across all firms per unit of space is then

$$\mathcal{L}_i^{k\Delta} = \int \frac{1}{k^2} N_{ji} \int_0^{\psi\sqrt{k^2/N_{ji}}} \left[\left(\frac{\varepsilon}{\varepsilon-1} \right)^{-\varepsilon} \frac{1}{W_i^{k\Delta}} \left(\frac{W_i^{k\Delta} T(\delta)}{B_i^{k\Delta} Z_j} \right)^{1-\varepsilon} c_i^{k\Delta} \left(P_i^{k\Delta} \right)^{\varepsilon} \mathcal{L}_i^{k\Delta} \right] \varpi \left(\frac{\delta}{\psi\sqrt{k^2/N_{ji}}} \right) 2\pi\delta d\delta dj$$

Using the change of variables $\tilde{\delta} = \delta/\Delta$ and $n_{ji} = \frac{\Delta^2}{k^2} N_{ji}$, this is

$$\begin{aligned} \mathcal{L}_i^{k\Delta} &= \int n_{ji} \int_0^{\psi/\sqrt{n_{ji}}} \left[\left(\frac{\varepsilon}{\varepsilon-1} \right)^{-\varepsilon} \frac{1}{W_i^{k\Delta}} \left(\frac{W_i^{k\Delta} t(\tilde{\delta})}{B_i^{k\Delta} Z_j} \right)^{1-\varepsilon} c_i^{k\Delta} \left(P_i^{k\Delta} \right)^{\varepsilon} \mathcal{L}_i^{k\Delta} \right] \varpi \left(\frac{\tilde{\delta}}{\psi/\sqrt{n_{ji}}} \right) 2\pi\tilde{\delta} d\tilde{\delta} dj \\ &= \int \left[\left(\frac{\varepsilon}{\varepsilon-1} \right)^{-\varepsilon} \frac{1}{W_i^{k\Delta}} \left(\frac{W_i^{k\Delta}}{B_i^{k\Delta} Z_j} \right)^{1-\varepsilon} c_i^{k\Delta} \left(P_i^{k\Delta} \right)^{\varepsilon} \mathcal{L}_i^{k\Delta} \right] \kappa(n_{ji}) dj \end{aligned}$$

Taking the limit as $\Delta \to 0$ and $k \to 0$ gives

$$\begin{aligned} \mathcal{L}_s &= \int \left[\left(\frac{\varepsilon}{\varepsilon-1} \right)^{-\varepsilon} \frac{1}{W_s} \left(\frac{W_s}{B_s Z_j} \right)^{1-\varepsilon} c_s \left(P_s \right)^{\varepsilon} \mathcal{L}_s \right] \kappa(n_{js}) dj \\ &= \left[\left(\frac{\varepsilon}{\varepsilon-1} \right)^{-\varepsilon} \frac{1}{W_s} \left(\frac{W_s}{B_s} \right)^{1-\varepsilon} c_s \left(P_s \right)^{\varepsilon} \mathcal{L}_s \right] \mathcal{Z}_s^{\varepsilon-1} \end{aligned}$$

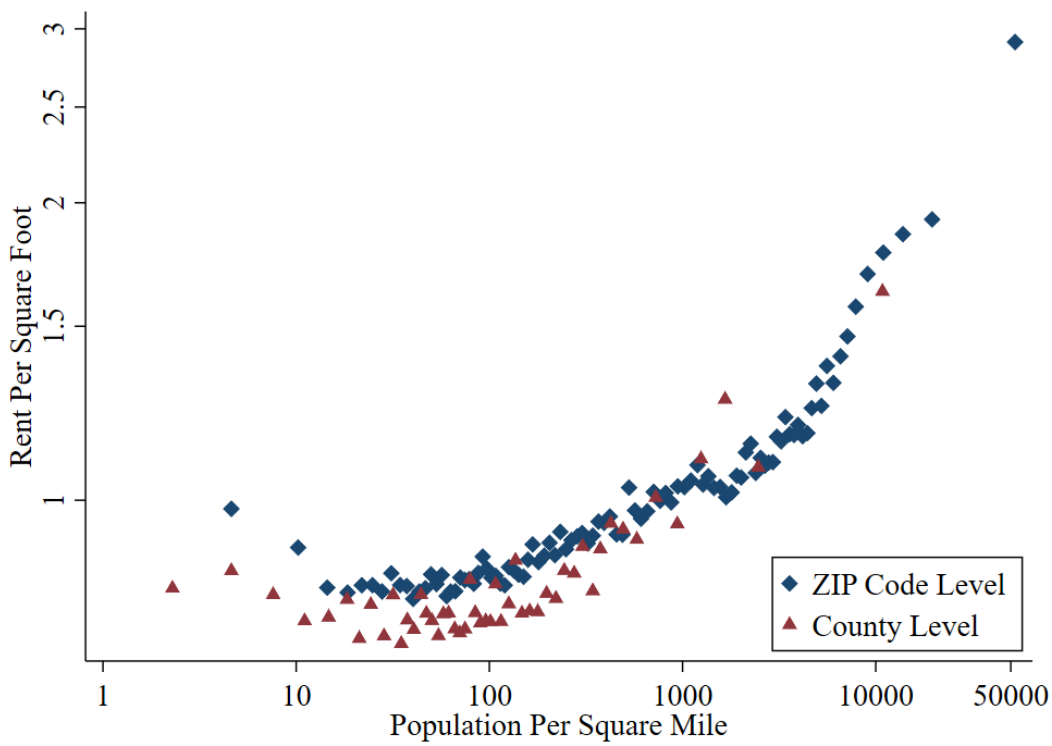
Combining this with the expression for the price level $P_s = \frac{\varepsilon}{\varepsilon-1} \frac{W_s}{B_s \mathcal{Z}_s}$ and simplifying gives

$$c_s = B_s \mathcal{Z}_s$$

B Rents and Density

Our theory relates the sorting of firms to local rent. Data on rent is incomplete and available at irregular geographic units, so our empirical exercises relate sorting to population density. Nevertheless, we can test empirically whether locations with higher population density have higher rent. We borrow rent data from Zillow for the year 2018. For population density, we use the 2012-2016 population estimate provided in the American Community Survey (ACS) dataset (Manson et al. (2021)). For each zipcode and county, Zillow provides an estimate of the rent per square foot. The rent per square foot is a preferable measure of rent than just the average rent in a location as the former controls for differences in housing size across locations, while the latter does not. Figure B.1 shows how the rent per square foot of a location, measured at either zip code or county levels, increases with the location population density.

Figure B.1: Rents across space



Notes: The figure presents the binned rent per square foot of a location (zipcode or county) in 2018 as a function of the location population density. Rent data comes from Zillow, while we use the 2012-2016 population estimate in the American Community Survey (ACS) from Manson et al. (2021) to construct the population density measure.

C Sorting

In this section we perform several robustness checks to our sorting results.

Table V presents the results of regressing the average of the log of the average employment density of each location, weighted by the number of establishments of a particular firm operating in a particular industry in the location, on the log of the national size of the firm and industry fixed effects. The first panel is the analogous to Figure 8 and presents the results as we vary M. Panels 2 to 7 use $M = 12$. The second panel restricts the analysis to firms with at least X plants. The third panel adds to the second panel a headquarters' location fixed effect for each firm. The fourth panel restricts the analysis to industries where there is a firm with at least X plants, and the fifth panel adds the fixed effect for the headquarters' location.

The sixth panel repeats the analysis by major industry. The last panel shows the robustness of the baseline results to excluding the own firm contribution to employment density, alternative weighting schemes, and to using only non-imputed data.

Table V: Sorting: Firm Size and Local Density

	(1) $\ln \bar{L}_j$	(2) $\ln \bar{L}_j$	(3) $\ln \bar{L}_j$	(4) $\ln \bar{L}_j$	(5) $\ln \bar{L}_j$
Baseline					
$\ln L_j$	0.222***	0.196***	0.165***	0.129***	0.0952***
	(0.00104)	(0.00101)	(0.000975)	(0.000926)	(0.000848)
Observations	3,645,763	3,665,497	3,670,994	3,672,721	3,673,053
R-squared	0.159	0.153	0.139	0.120	0.099
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	3	6	12	24	48
Firms with at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln L_j$	0.165***	0.134***	0.132***	0.135***	0.146***
	(0.000975)	(0.00655)	(0.00982)	(0.0153)	(0.0249)
Observations	3,670,994	11,203	4,904	1,892	876
R-squared	0.139	0.385	0.391	0.405	0.384
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
Controlling for HQ location, firms with at least X plants	*X = 2*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln L_j$	0.0229***	0.0347***	0.0396***	0.0516***	0.0791**
	(0.00171)	(0.00563)	(0.00863)	(0.0159)	(0.0350)
Observations	145,186	9,700	4,182	1,534	652
R-squared	0.705	0.665	0.676	0.693	0.664
SIC8 FE	Yes	Yes	Yes	Yes	Yes
HQ Location FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
Industries where largest firm has at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln L_j$	0.165***	0.178***	0.176***	0.176***	0.172***
	(0.000975)	(0.00109)	(0.00119)	(0.00140)	(0.00164)
Observations	3,670,994	2,861,609	2,424,907	1,829,818	1,387,742
R-squared	0.139	0.114	0.102	0.091	0.080
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
Controlling for HQ location, industries where largest firm has at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln L_j$	0.0229***	0.0252***	0.0244***	0.0222***	0.0201***
	(0.00171)	(0.00179)	(0.00190)	(0.00214)	(0.00244)
Observations	145,186	124,065	106,163	81,372	62,957
R-squared	0.705	0.711	0.720	0.729	0.735
SIC8 FE	Yes	Yes	Yes	Yes	Yes
HQ Location FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
	Baseline	*Excluding Own Contribution*	*Unweighted*	*Weighted by Employment*	*Non-imputed*
$\ln L_j$	0.165***	0.164***	0.161***	0.164***	0.202***
	(0.000975)	(0.000975)	(0.000974)	(0.000981)	(0.00111)
Observations	3,670,994	3,668,734	3,670,994	3,670,994	2,605,050
R-squared	0.139	0.138	0.138	0.139	0.169
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Notes: The table presents the results of regressing the log of the average employment density across all of the firm plants on the log employment of the firm at the national level and industry fixed effects. The first panel presents the results as we vary M. Panels 2 to 7 use $M = 12$. The second panel restricts the analysis to firms with at least X plants. The third panel adds to the second panel a headquarters' location fixed effect for each firm. The fourth panel restricts the analysis to industries where there is a firm with at least X plants, and the fifth panel adds the fixed effect for the headquarters' location. The sixth panel repeats the analysis by major industry. The last panel shows the robustness of the baseline results to excluding the own firm contribution to employment density, alternative weighting schemes, and to using only non-imputed data.

D The Largest Firm in Town

Table VI presents the results of regressing the log of the national size of the firm with most plants in a location, $L_{j^*(s)}$, on the log of the employment density of the location, $\mathcal{L}_s$, and industry fixed effects. If there is a tie in the identity of the firm with most plants in a location, we take the average of the log national employment of the firms. The first panel presents our baseline results for different spatial resolutions M. The second panel repeats the analysis but restricting to firms with at least X plants, and the third panel restricts the analysis to industries where the largest firm has at least X plants. The fourth panel presents the results by major industry. The fifth panel presents the results when excluding the firm's own contribution to a location employment, when using alternative ways to resolving ties (in terms of which firm has the highest amount of plants in a location), and when using only non-imputed data.

Table VI: The national size of the largest firm in town

	(1) $\ln L_{j^*(s)}$	(2) $\ln L_{j^*(s)}$	(3) $\ln L_{j^*(s)}$	(4) $\ln L_{j^*(s)}$	(5) $\ln L_{j^*(s)}$
Baseline					
$\ln \mathcal{L}_s$	0.194***	0.293***	0.395***	0.486***	0.594***
	(0.00169)	(0.00206)	(0.00266)	(0.00350)	(0.00461)
Observations	3,131,324	2,551,226	1,984,474	1,473,278	1,006,305
R-squared	0.593	0.608	0.616	0.630	0.644
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	3	6	12	24	48
Firms with at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln \mathcal{L}_s$	0.395***	0.239***	0.191***	0.158***	0.131***
	(0.00266)	(0.00620)	(0.00659)	(0.00729)	(0.00811)
Observations	1,984,474	356,238	308,839	253,996	211,517
R-squared	0.616	0.561	0.564	0.560	0.554
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
Industries where largest firm has at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln \mathcal{L}_s$	0.395***	0.411***	0.430***	0.471***	0.516***
	(0.00266)	(0.00276)	(0.00291)	(0.00325)	(0.00366)
Observations	1,984,474	1,390,883	1,125,690	813,539	616,248
R-squared	0.616	0.613	0.609	0.605	0.600
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
By major industry	*Baseline*	*Manufacturing*	*Services*	*Retail Trade*	*FIRE*
$\ln \mathcal{L}_s$	0.395***	0.0746***	0.279***	0.562***	0.557***
	(0.00266)	(0.00549)	(0.00489)	(0.00491)	(0.00729)
Observations	1,984,474	245,343	647,569	421,352	133,956
R-squared	0.616	0.256	0.347	0.679	0.557
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
	Baseline	*Excluding Own Contribution*	*Discarding Ties*	*Largest Firm Among Ties*	*Non-imputed*
$\ln \mathcal{L}_s$	0.395***	0.372***	0.617***	0.694***	0.404***
	(0.00266)	(0.00491)	(0.00386)	(0.00276)	(0.00288)
Observations	1,984,474	568,124	1,449,578	1,984,474	1,666,909
R-squared	0.616	0.550	0.604	0.585	0.618
SIC8 FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Notes: The table presents the results of regressing the log of the national size of the firm with most plants in a location on the log of the employment density of the location and industry fixed effects. The first panel presents the results as we vary M. Panels 2 to 5 use $M = 12$. The second panel presents the results for firms with at least X plants. The third panel presents the results for industries with at least one firm with X plants. The fourth panel presents the results by major industry. The fifth panel presents the results when excluding the firm's own contribution to employment in a location, when discarding locations where there is a tie in the identity of the firm with most plants or using the size of the largest firm in this case, and when using only non-imputed data.

E Span of Control

In this section we present several robustness checks to our span-of-control results. Table VII presents the results of regressing the log of the average plant employment of a firm within a location on the firm's log national employment, and controlling for the firm's log number of plants in the location and square of the log number of plants in the location. In all cases, we subtract the own firm contribution of employment in a location from that firm's total employment. The first panel presents the results for different values of M. The second panel restricts the analysis to firms with at least X plants, while the third panel restricts the analysis to industries where there is one firm with at least X plants. The fourth panel presents the results by major industry. Table VIII presents some additional robustness results. The first panel presents the regression results without subtracting the own firm contribution of employment in a location from that firm's total employment. Panels 2 and 3 subtract the own firm contribution of employment in a location from that firm's total employment. Panel 2 adds higher order terms of the log of the number of establishments of a firm in a location as controls, while Panel 3 restricts attention to non-imputed data.

Table VII: Span of Control

	(1)	(2)	(3)	(4)	(5)
	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$
Baseline					
$\ln L_{j,-js}$	0.0925***	0.103***	0.114***	0.123***	0.131***
	(0.00101)	(0.000918)	(0.000897)	(0.000911)	(0.000962)
$\ln n_{js}$	0.131***	0.109***	0.137***	0.163***	0.172***
	(0.0171)	(0.0114)	(0.00897)	(0.00753)	(0.00682)
$(\ln n_{js})^2$	0.00245	-0.0502***	-0.0813***	-0.0852***	-0.0811***
	(0.0135)	(0.00715)	(0.00447)	(0.00316)	(0.00251)
Observations	311,244	376,723	409,364	408,521	386,094
R-squared	0.632	0.608	0.573	0.542	0.511
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	3	6	12	24	48
Firms with at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln L_{j,-js}$	0.114***	0.154***	0.187***	0.233***	0.275***
	(0.000897)	(0.00157)	(0.00190)	(0.00241)	(0.00296)
$\ln n_{js}$	0.137***	-0.0121	-0.0765***	-0.132***	-0.168***
	(0.00897)	(0.00960)	(0.00990)	(0.0104)	(0.0111)
$(\ln n_{js})^2$	-0.0813***	-0.0339***	-0.0188***	-0.00932*	-0.00158
	(0.00447)	(0.00457)	(0.00465)	(0.00494)	(0.00528)
Observations	409,364	233,744	197,273	157,980	126,999
R-squared	0.573	0.658	0.686	0.714	0.746
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
Industries where largest firm has at least X plants	*Baseline*	*X = 10*	*X = 20*	*X = 50*	*X = 100*
$\ln L_{j,-js}$	0.114***	0.114***	0.112***	0.108***	0.104***
	(0.000897)	(0.000897)	(0.000898)	(0.000901)	(0.000911)
$\ln n_{js}$	0.137***	0.133***	0.127***	0.102***	0.0720***
	(0.00897)	(0.00895)	(0.00894)	(0.00888)	(0.00883)
$(\ln n_{js})^2$	-0.0813***	-0.0797***	-0.0768***	-0.0673***	-0.0564***
	(0.00447)	(0.00446)	(0.00444)	(0.00438)	(0.00431)
Observations	409,364	405,623	394,807	369,321	336,424
R-squared	0.573	0.573	0.574	0.577	0.588
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12
By major industry	*Baseline*	*Manufacturing*	*Services*	*Retail Trade*	*FIRE*
$\ln L_{j,-js}$	0.114***	0.154***	0.130***	0.124***	0.0627***
	(0.000897)	(0.0104)	(0.00214)	(0.00118)	(0.00180)
$\ln n_{js}$	0.137***	1.127***	0.325***	-0.0506***	0.126***
	(0.00897)	(0.145)	(0.0281)	(0.0109)	(0.0159)
$(\ln n_{js})^2$	-0.0813***	-0.252**	-0.121***	-0.0451***	-0.0398***
	(0.00447)	(0.102)	(0.0164)	(0.00544)	(0.00720)
Observations	409,364	8,864	95,301	164,941	84,798
R-squared	0.573	0.541	0.506	0.665	0.442
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	12	12	12	12	12

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Notes: The table presents the results of regressing the log of the average plant employment of a firm within a location on the log national employment of the firm (excluding the own firm contribution of employment in a location from that firm's total employment), industry fixed effects and controls for the number of plants that the firm has in the location. The first panel presents the results for different values of M. Panels 2 to 4 use $M = 12$. The second panel restricts the analysis to firms with at least X plants, while the third panel restricts the analysis to industries where there is one firm with at least X plants. The fourth panel presents the results by major industry. In all cases, we subtract the own firm contribution of employment in a location from that firm's total employment.

Table VIII: Span of Control: Additional Exercises

	(1)	(2)	(3)	(4)	(5)
	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$	$\ln \bar{l}_{js}$
$\ln L_j$	0.216***	0.240***	0.277***	0.318***	0.371***
	(0.000531)	(0.000495)	(0.000501)	(0.000528)	(0.000571)
$\ln n_{js}$	-0.367***	-0.548***	-0.636***	-0.714***	-0.814***
	(0.0156)	(0.00924)	(0.00679)	(0.00558)	(0.00486)
$(\ln n_{js})^2$	0.0807***	0.0737***	0.0544***	0.0560***	0.0605***
	(0.0131)	(0.00598)	(0.00341)	(0.00240)	(0.00187)
Observations	2,033,197	2,574,018	3,009,650	3,343,237	3,592,041
R-squared	0.597	0.568	0.550	0.544	0.552
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	3	6	12	24	48
$\ln L_{j,-js}$	0.0925***	0.103***	0.114***	0.123***	0.131***
	(0.00101)	(0.000918)	(0.000897)	(0.000911)	(0.000963)
$\ln n_{js}$	-0.0310	0.206***	0.319***	0.319***	0.339***
	(0.0497)	(0.0335)	(0.0256)	(0.0211)	(0.0186)
$(\ln n_{js})^2$	0.268***	-0.211***	-0.344***	-0.278***	-0.274***
	(0.0890)	(0.0570)	(0.0398)	(0.0297)	(0.0235)
$(\ln n_{js})^3$	-0.102**	0.0688**	0.0953***	0.0575***	0.0559***
	(0.0430)	(0.0278)	(0.0183)	(0.0125)	(0.00901)
$(\ln n_{js})^4$	0.00909	-0.00757**	-0.00840***	-0.00367**	-0.00385***
	(0.00558)	(0.00385)	(0.00248)	(0.00156)	(0.00103)
Observations	311,244	376,723	409,364	408,521	386,094
R-squared	0.632	0.608	0.573	0.542	0.512
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	3	6	12	24	48
Non-Imputed					
$\ln L_{j,-js}$	0.0854***	0.0947***	0.106***	0.115***	0.125***
	(0.00114)	(0.00104)	(0.00101)	(0.00102)	(0.00107)
$\ln n_{js}$	0.193***	0.153***	0.154***	0.174***	0.173***
	(0.0208)	(0.0138)	(0.0104)	(0.00863)	(0.00770)
$(\ln n_{js})^2$	-0.0165	-0.0546***	-0.0777***	-0.0830***	-0.0795***
	(0.0169)	(0.00898)	(0.00527)	(0.00372)	(0.00294)
Observations	227,058	280,597	310,989	317,407	306,306
R-squared	0.647	0.625	0.592	0.563	0.535
SIC8-location FE	Yes	Yes	Yes	Yes	Yes
M	3	6	12	24	48

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Notes: The table presents the results of regressing the log of the average plant employment of a firm within a location on the log national employment of the firm, industry fixed effects and controls for the number of plants that the firm has in the location. The first panel presents the results for different values of M, without excluding the own firm contribution of employment in a location from that firm's total employment. Panels 2 and 3 exclude the own firm contribution of employment in a location from that firm's total employment. Panel 2 includes higher order terms of the log of the number of establishments of a firm in a location as control variables. Panel 3 restricts attention to non-imputed data.

F Numerical Exploration: Algorithm

In this section we describe the algorithm that we used to solve for the industry equilibrium. Our algorithm exploits the first order conditions of the firm's problem (equations 4 and 5),

$$x_s z_j^{\varepsilon-1}\kappa'(n_{js}) \leq R_s + \lambda_j, \qquad \text{with equality if } n_{js} > 0, \text{ and}$$
$$\lambda_j = -\frac{d[z(q_j,N_j)^{\varepsilon-1}]}{dN_j}\int_s x_s \kappa(n_{js})ds,$$

where $z_j = z(q_j, N_j)$ with $N_j = \int_s n_{js}ds$, $x_s = \frac{I_s/(\varepsilon-1)}{\mathcal{Z}_s^{\varepsilon-1}}$ with $\mathcal{Z}_s = \left(\int_j z_j^{\varepsilon-1}\kappa(n_{js})dj\right)^{\frac{1}{\varepsilon-1}}$, and $R_s = R(I_s)$.

Our algorithm iterates on three univariate functions, $\mathcal{Z}_s \,\forall\, s$, and $\{N_j, \lambda_j\} \,\forall\, j$. Let $t = 0, 1, 2, ...$ denote the iteration round. Given an initial guess or the results of the previous iteration, $\mathcal{Z}_s^{t-1} \,\forall\, s$, and $\{N_j^{t-1}, \lambda_j^{t-1}\} \,\forall\, j$, we can compute the following objects: (i) $n_{j,s}^t \,\forall\, j, s$ (using equation 4), (ii) $N_j^t = \int_s n_{js}^t ds$, (iii) $z_j^t = z(q_j, N_j^t)$, (iv) $\mathcal{Z}_s^t = \left(\int_j \left(z_j^t\right)^{\varepsilon-1}\kappa(n_{js}^t)dj\right)^{\frac{1}{\varepsilon-1}}$, (v) $x_s^t = \frac{I_s}{(\varepsilon-1)(\mathcal{Z}_s^t)^{\varepsilon-1}}$, and (vi) $\lambda_j^t = -\int_s \frac{\partial z(q_j,N_j^t)^{\varepsilon-1}}{\partial N} x_s^t \kappa(n_{js}^t)ds$. We repeat this procedure until a convergence criterion is satisfied, that is, until there is a $t = \tilde{t}$ such that

$$||\mathcal{Z}_s^{\tilde{t}} - \mathcal{Z}_s^{\tilde{t}-1}|| + ||N_j^{\tilde{t}} - N_j^{\tilde{t}-1}|| + ||\lambda_j^{\tilde{t}} - \lambda_j^{\tilde{t}-1}|| \leq \epsilon \ ,$$

where $||\cdot||$ is the sup norm and ϵ is a small number.

We use a two dimensional grid of points to numerically integrate when necessary and to evaluate the convergence criterion. Specifically, we use a two dimensional grid of S locations and J firms. For each iteration, a sufficient state is the value of the functions N_j, λ_j, and $\mathcal{Z}_s^t$, at these grid points. For each point j, s on the grid, we require only the values of $\mathcal{Z}_s^{t-1}$, N_j^{t-1}, and λ_j^{t-1} to evaluate n_{js}^t. To find N_j^t, numerically integrate across the locations using the trapezoid rule and the values of n_{js}^t at each of the S location grid points. Similarly, to find λ_j^{t-1}, we numerically integrate across locations. To find each location's local productivity, for any s, we numerically integrate across firms using the values of z_j^t and n_{js}^t at each grid point. This delivers new values of the functions at each of the $S \times J$ grid points. Finally, to evaluate the norms, we evaluate the convergence criterion by numerically integrating using the grid points.

In our numerical simulation, we used $J = 50$ and $S = 100$, but we found no noticeable difference in the solution when we used a grid of $J = 30$ and $S = 50$.

A complication arises due to the fact that $\kappa(n)$ is linear in the neighborhood of $n = 0$ (see Lemma 3). Because of this linearity, n_{js} move quite a bit across iterations in response to very small changes in $\mathcal{Z}_s$, N_j and λ_j. This can generate cycles in the iteration process. We handle this issue in two ways.

First, at each iteration, we do not fully update the policy functions. That is, we evaluate iteration $t+1$ using $\tilde{\mathcal{Z}}_s^t$, $\tilde{N}_j^t$ and $\tilde{\lambda}_j^t$ instead of $\mathcal{Z}_s^t$, N_j^t and λ_j^t, where $\tilde{\mathcal{Z}}_s^t = \varsigma\mathcal{Z}_s^{t-1} + (1-\varsigma)\mathcal{Z}_s^t$, $\tilde{N}_j^t = \varsigma N_j^{t-1} + (1-\varsigma)N_j^t$, and $\tilde{\lambda}_j^t = \varsigma\lambda_j^{t-1} + (1-\varsigma)\lambda_j^t$, where $\varsigma \in (0,1)$ is a dampening parameter. In principle, there exists a $\varsigma < 1$ such that cycles are not a concern. However, in many situations (i.e. sets of parameter values) the low degree of updating of the policy functions makes the code extremely slow.[44] Thus, we take an additional step.

Second, we replace the function $\kappa(n)$ with

$$\hat{\kappa}(n) = \alpha\mathcal{H}(n) + (1-\alpha)\kappa(n) \ ,$$

where $\mathcal{H}(n) = 1 - e^{-n/h}$. Notice that $\mathcal{H}'(n) > 0$, $\mathcal{H}''(n) < 0$, with $\mathcal{H}(0) = 0$, $\lim_{n\to\infty}\mathcal{H}(n) = 1$.[45] As a result, $\hat{\kappa}'(n) > 0$, $\hat{\kappa}''(n) < 0$, with $\hat{\kappa}(0) = 0$, $\lim_{n\to\infty}\hat{\kappa}(n) = 1$.

That is, in the iteration process we use $\hat{\kappa}(n)$ instead of $\kappa(n)$. In our experiments, a combination of $\varsigma > 0$ and $\alpha > 0$ are able to handle cycles and thus allows the code to converge quickly. For the numerical explorations presented in this paper we use $\varsigma = 0.97$ and $\alpha = 0.0001$.

To ensure that this approximation yields an accurate solution, we can evaluate whether the resulting policy function found using $\hat{\kappa}$ is the solution to each firm's true problem that uses κ. Let $\hat{\mathcal{Z}}_s$, $\hat{N}_j$ and $\hat{\lambda}_j$ denote the solution of the iteration process (i.e. once the convergence criterion is satisfied) when we solve the firms problem using $\hat{\kappa}(n)$. We can also easily obtain $\hat{n}_{js}$, $\hat{z}_j = z(q_j, \hat{N}_j)$, and $\hat{x}_s$. To gauge the accuracy of the approximate solution, we compute,

$$\begin{aligned}
\text{absolute error } &= \int_s\int_j \mathbf{1}\left[\hat{n}_{js} > 0\right] \overbrace{\left[\hat{x}_s\hat{z}_j^{\varepsilon-1}\kappa'(\hat{n}_{js}) - (R_s + \hat{\lambda}_j)\right]}^{\text{error in equation 4}} dj ds \ , \\
\text{relative error } &= \frac{\int_s\int_j \mathbf{1}\left[\hat{n}_{js} > 0\right]\left[\hat{x}_s\hat{z}_j^{\varepsilon-1}\kappa'(\hat{n}_{js}) - (R_s + \hat{\lambda}_j)\right] dj ds}{\int_s\int_j \mathbf{1}\left[\hat{n}_{js} > 0\right](R_s + \hat{\lambda}_j) dj ds} \ .
\end{aligned}$$

That is, the first expression computes the absolute error of the allocation using $\hat{\kappa}(n)$, but evaluating the first order condition using $\kappa(n)$, while the second expression provides the absolute error, relative to the level of costs for firm j in location s, as described by the RHS of equation 4. For our baseline equilibrium, we find that absolute error $= 0.00008$, and relative error $= 0.000025$. That is, the absolute error is 0.0025% of the average level of the RHS of the first order condition. This provides reassurance that the solution under $\hat{\kappa}(n)$ is a good approximation of the actual solution.

G Numerical Computation of Bounds $\bar{\pi}_j^{k\Delta}$ and $\underline{\pi}_j^{k\Delta}$

In this section we provide a numerical example of the upper and lower bounds $\bar{\pi}_j^{k\Delta}$ and $\underline{\pi}_j^{k\Delta}$ for any given k and Δ. We then use this example to discuss how the gap between these bounds changes with Δ.

We begin by noticing that the expression for the upper bound $\bar{\pi}_j^{k\Delta}$ presented in Claim A.11 can be

[44] For high enough values of ς the code can take many hours to converge, even when J and S are small.

[45] The parameter h allows us to modify the concavity of the function $\mathcal{H}(n)$. Here, we used $h = 0.01$.

written as

$$\bar{\pi}_j^{k\Delta} \equiv \sup_{\{n_i \geq 0\}} \sum_{i \in I^k} -n_i k^2 \underline{R}_i^k \xi + z\left(q_j, \sum_{i \in I^k} n_i k^2\right)^{\varepsilon-1} \sum_{i \in I^k} \bar{D}_i^k \bar{b}_i^k k^2 \kappa(n_i) + z\left(q_j, 0\right)^{\varepsilon-1} \bar{D}\bar{b}k^{-1} \int_0^k t\left(\frac{\delta}{\Delta}\right)^{1-\varepsilon} d\delta \ .$$

A similar expression can be obtained for $\underline{\pi}_j^{k\Delta}$. We can obtain expressions for n_i, the number of plants in square $\mathcal{S}_i^k$, from the first order condition,

$$z\left(q_j, \sum_{i \in I^k} n_i k^2\right)^{\varepsilon-1} \bar{D}_i^k \bar{b}_i^k \kappa'(n_i) = \underline{R}_i^k - \frac{d\left[z\left(q_j, \sum_{i \in I^k} n_i k^2\right)^{\varepsilon-1}\right]}{dn_i} \sum_{i \in I^k} \bar{D}_i^k \bar{b}_i^k \kappa(n_i) \ .$$

Again, a similar first-order condition can be produced for the number of establishments in each location implied by the lower bound.

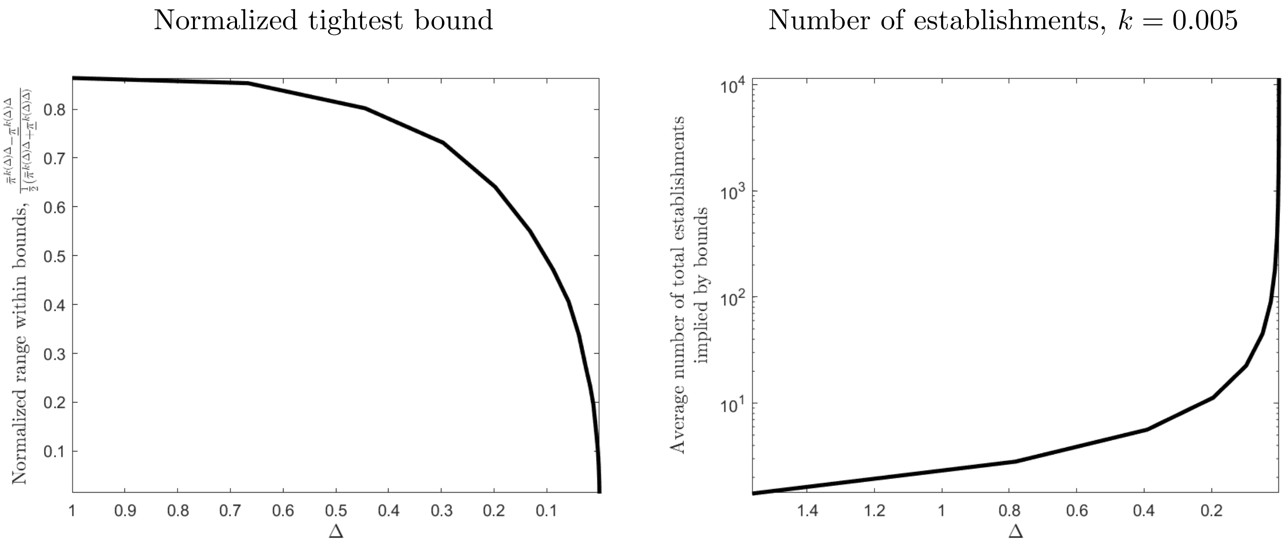
For given vectors $\{\bar{D}^k, \bar{b}^k, \underline{R}^k\}_{i \in I^k}$ and $\{\underline{D}^k, \underline{b}^k, \bar{R}^k\}_{i \in I^k}$, we use the first order conditions to obtain $\bar{n}_i$ and $\underline{n}_i$ for firm j, the solutions to the two maximization problems. To do this, notice that the first order condition closely resembles that one presented in (4), and thus we exploit the procedure presented in Appendix F. With expressions for $\bar{n}_i$ and $\underline{n}_i$ we can then readily compute $\bar{\pi}_j^{k\Delta}$ and $\underline{\pi}_j^{k\Delta}$.

For the example presented below, we restrict our attention to space defined in a unit square, $\mathcal{S} = [0,1]^2$, and we assume that for any $\{s_1, s_2\} \in \mathcal{S}$ we have that $b_s = 50(1+s_1)(1+2s_2)$, $D_s = 20(1+s_1)(1+2s_2)$, and $R_s = 0.1(1+s_1)(1+s_2)$. Further, we set $\varepsilon = 2$, $q_j = 1$, and we follow Section 3.1 and assume that $t(\delta/\sqrt{\phi}) = e^{\delta/\sqrt{\phi}}$ with $\phi = 0.04$. Likewise, we set $z(q,N) = qe^{-N/\sigma}$ with $\sigma = 5$.

For a given k, we divide $\mathcal{S}$ into squares with side of length k. We do this for various values of k ranging from 1/2 to 5/1000. Because the number of partitions equals $(1/k)^2$, the number of squares in I^k ranges from 4 to 40,000. Then, for a fixed value of Δ, let $k(\Delta)$ denote which partition provides the tightest normalized range within the upper and lower bounds, $k(\Delta) = \arg\min_k \left(\bar{\pi}_j^{k\Delta} - \underline{\pi}_j^{k\Delta}\right) / \left[\frac{1}{2}\left(\bar{\pi}_j^{k\Delta} + \underline{\pi}_j^{k\Delta}\right)\right]$. The function $\left(\bar{\pi}_j^{k(\Delta)\Delta} - \underline{\pi}_j^{k(\Delta)\Delta}\right) / \left[\frac{1}{2}\left(\bar{\pi}_j^{k(\Delta)\Delta} + \underline{\pi}_j^{k(\Delta)\Delta}\right)\right]$ provides a good notion of the tightness of the bounds.

The left panel of Figure G.1 presents the normalized tightest bounds for the example we explore in this section, and the right panel presents the (log of the) average total number of establishments implied by the upper and lower bounds for the case $k = 0.005$. The figure shows that the bounds are not particularly tight when Δ is large, but they tighten as Δ falls. Consistent with our theoretical results, the bounds become very tight as Δ approaches zero. Likewise, the results presented in the right panel suggest that the bounds are tight when Δ is such that firms operate many plants, but less tight when Δ is large and the firm operates few plants. For example, if the firm were to operate over 1,000 establishments, the bounds appear to be very tight with a normalized gap close to zero. However, if the firm operates 10 total establishments, the normalized gap is around 0.7.

Figure G.1: Bounds



H A discrete example with firms with few plants

Our approach to the problem of choosing plant locations was to focus on the limiting economy as Δ approaches zero. In this limit, the problem admits an analytical solution that we use to derive theoretical predictions that we then corroborate empirically. While our theoretical results regarding uniform convergence of the policy function are reassuring of the relevance of these predictions for industries for which the limit is a good approximation, they may be less relevant for industries in which plants tend to have large catchment areas.

In this section, we use two numerical examples to explore firm choices outside of the limit, i.e. large Δ. In both examples, firms choose to have a small number of plants. In the first example, Δ is very large and firms choose to have either one or two plants, as transport costs are low. In this example, we run the same regressions as in the main text of the paper and fail to detect sorting. In the second example, we lower Δ and solve for the resulting plant configuration. Now firms place more plants across locations. In this example, we detect sorting that is consistent with the theoretical predictions of the limiting economy.

To make the numerical exercise feasible and operational we make a set of modifications to our model: (i) we follow Tintelnot (2016) and assume that a firm produces a continuum of goods, where each location that may be used to produce the good has a different idiosyncratic cost of producing a particular good, and (ii) we assume that there is only a discrete, and small, set of feasible locations. Modification (ii) allows us to use the toolkit in Arkolakis et al. (2017) to solve the plant location problem for each firm. Modification (i) increases the number of configurations that can be ruled out before resorting to evaluating all remaining combinations (the brute force approach).

Consider a set of discrete locations, where s denotes a location. Every firm produces a continuum of goods, $i \in [0,1]$. For each good, each location where the firm places a plant o has an idiosyncratic cost shifter a_{oi} with $\Pr[a_{oi} > a] = e^{-a^{\theta}}$, so that the unit cost of supplying good i from a plant in location o to location s is $\frac{w}{B_o Z} a_{oi} T_{os}$. Thus, the minimal cost to the firm of supplying good i to location s is $\lambda_{is} = \min_{o \in O} \frac{w}{B_o Z} a_{oi} T_{os}$, with associated distribution $\Pr(\lambda_{is} > \lambda) = e^{-\lambda^{\theta} \sum_{o \in O} \left(\frac{B_o Z}{w T_{os}}\right)^{\theta}}$.

Demand for each good at each location s is $D_s p^{-\varepsilon}$, which implies a markup of $\frac{\varepsilon}{\varepsilon - 1} > 1$. Given a set of active plants O, a firm's total profits are

$$\pi(O,Z) = \left(\frac{1}{\varepsilon - 1}\right)\left(\frac{\varepsilon}{\varepsilon - 1}\right)^{-\varepsilon} \sum_s D_s \int_0^1 \lambda_{is}^{1-\varepsilon} di - \sum_{o \in O} R_o \xi \ ,$$

where, given the distribution for λ_{is}, $\int_0^1 \lambda_{is}^{1-\varepsilon} di = \Gamma\left(1 - \frac{\varepsilon-1}{\theta}\right)\left[\sum_{o \in o}\left(\frac{B_o Z}{w T_{os}}\right)^{\theta}\right]^{\frac{\varepsilon-1}{\theta}}$. Therefore, a firm's profits from operating a set of plants O are given by

$$\pi(O,Z) = \left(\frac{1}{\varepsilon - 1}\right)\left(\frac{\varepsilon}{\varepsilon - 1}\right)^{-\varepsilon} \Gamma\left(1 - \frac{\varepsilon - 1}{\theta}\right) \sum_s D_s \left[\sum_{o \in O}\left(\frac{B_o Z}{w T_{os}}\right)^{\theta}\right]^{\frac{\varepsilon-1}{\theta}} - \sum_{o \in O} R_o \xi \ ,$$

where plant employment in location o is given by

$$l_o = \frac{\varepsilon}{w}\left(\frac{\varepsilon}{\varepsilon-1}\right)^{-\varepsilon} \Gamma\left(1 - \frac{\varepsilon-1}{\theta}\right) \sum_s D_s \left[\sum_{\tilde{o} \in O}\left(\frac{B_{\tilde{o}} Z}{w T_{\tilde{o}s}}\right)^{\theta}\right]^{\frac{\varepsilon-1}{\theta}} \frac{\left(\frac{B_o Z}{w T_{os}}\right)^{\theta}}{\sum_{\tilde{o} \in O}\left(\frac{B_{\tilde{o}} Z}{w T_{\tilde{o}s}}\right)^{\theta}}.$$

Notice that we recover the specification in the main text, where each location in space is served by only one the firm's plants, when $\theta \to \infty$.[46] At each location s a firm with productivity q sells goods at price

$$p_s(q) = \frac{\varepsilon}{\varepsilon - 1}\left[\sum_{o \in O(q)}\left(\frac{B_o Z(q, N(q))}{w T_{os}}\right)^{\theta}\right]^{-\frac{1}{\theta}} .$$

As a result, the price index in location s is given by $P_s = \left(\sum_q [p_s(q)]^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$.

For the numerical exploration of the model, we assume that space is characterized by an evenly spaced grid dividing the unit square into $\overline{N}^2$ smaller squares (i.e., an $\overline{N} \times \overline{N}$ grid). The center of each square is

[46] It is straightforward to show how we recover our specification when θ diverges to infinity,

$$\lim_{\theta \to \infty}\left[\sum_{o \in O}\left(\frac{B_o Z}{w T_{os}}\right)^{\theta}\right]^{\frac{\varepsilon-1}{\theta}} = \lim_{\theta \to \infty}\left[\sum_{o \in O}\left(\left(\frac{B_o Z}{w T_{os}}\right)^{\varepsilon-1}\right)^{\frac{\theta}{\varepsilon-1}}\right]^{\frac{\varepsilon-1}{\theta}} = \max_{o \in O}\left(\frac{B_o Z}{w T_{os}}\right)^{\varepsilon-1}.$$

a location. Plants can be set up at all locations. In our example, $\overline{N}^2 = 36$. This results in more than 68 billion (2^{36}) possible permutations of the allocation of plants across space. We have been unable to consistently solve numerically cases (for any parameterization) with a larger set of potential locations. With more locations, locations get closer to each other and thus are "more similar to each other"— given that fundamentals are drawn from continuous distributions. Hence, the pruning approach in Arkolakis et al. (2017) cannot easily eliminate dominated plant configurations, and thus one must resort to an approach that considers all of the potential configurations, which becomes quickly infeasible as the number of locations expands.

We parameterize the model as follows. We let $\varepsilon = 2$, $\theta = 2$, $w = 1$, $B_o = 1 \; \forall \; o$, $R_s = \frac{3}{2}\sin((3\pi x)(\pi y)) + \frac{3}{2}$, $D_s = -\frac{3}{2}\sin((3\pi x)(\pi y)) + \frac{3}{2}$. Further, $T_{os} = \frac{1+\delta_{os}}{\Delta}$, $Z(q, N) = \frac{q}{1+(\Delta^2 N)^{0.45}}$, and $\xi = \Delta^2$. Finally, we solve the plant location problem for 15 firm types, with intrinsic productivity levels $q \in \{3, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25\}$, and we study the resulting plant allocations for $\Delta \in \{9, 1\}$.

Table IX compares the number of plants, $n^*(q)$, chosen by firms of different productivity, q, as we vary Δ. When $\Delta = 9$, all firms set up only one plant, with the exception of firms with the highest productivity levels which set up two plants. Also, all firms place one plant at the same location, the location that minimizes production costs. This follows from the fact that for high Δ, transportation costs are low regardless of distance, so most firms find it optimal to serve customers with just one plant. For $\Delta = 1$ transport costs increase, and firms place substantially more plants across space in order to save on transport costs. Now, we observe substantial variation in the number of plants across firms.

Table IX: Number of plants per firm, across Δ

	q	3	4	5	7	9	10	12	13	15	17	18	20	22	23	25
Δ	9	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2
	1	9	9	10	13	15	15	17	18	19	20	20	20	22	22	22

We now replicate the empirical analysis in the main body of the paper. We first divide our $\overline{N} \times \overline{N}$ grid on the unit square into $\mathcal{S}_M = \overline{N}^2/M^2$ sub-squares, each sub-square consisting of an $M \times M$ grid. Within each sub-square, i, let $n_i^*(q)$ be the number of active plants operated by a firm with productivity q, where $\sum_{i=1}^{\mathcal{S}_M} n_i^*(q) = N^*(q)$ denotes the total number of active plants for that firm across all locations. Also, let $\overline{\mathcal{L}}_i$ represent average population in sub-square i, $\overline{\mathcal{L}}_i = (1/M^2)\sum_{s\in i} \frac{D_s P_s^{-\varepsilon}}{w}$.[47] As in the main body of the paper, we use these objects to construct a measure of average density across locations for a firm with productivity

[47]We have that demand at location s satisfies $D_s = wL_s P_s^{\varepsilon}$, which implies that population at location s is given by $L_s = \frac{D_s P_s^{-\varepsilon}}{w}$.

q as,

$$\overline{L}(q) = \sum_{i=1}^{\mathcal{S}_M} \frac{n_i^*(q)}{N^*(q)} \times \overline{\mathcal{L}}_i \ .$$

Finally, we define total firm employment as $L(q) = \sum_s l_s(q)$, where $l_s(q)$ is the employment of an active plant of a firm with productivity q at location s.

Table X: Sorting: Firm Size and Local Density in a solved example

	$\Delta = 9$	$\Delta = 9$	$\Delta = 1$	$\Delta = 1$
	(1)	(2)	(3)	(4)
	$\ln \bar{L}(q)$	$\ln \bar{L}(q)$	$\ln \bar{L}(q)$	$\ln \bar{L}(q)$
$\ln L\,(q)$	0.0150	-0.0343	0.0167***	0.0172**
	(0.0093)	(0.0215)	(0.005)	(0.0048)
Observations	15	15	15	15
R-squared	0.180	0.180	0.429	0.473
M	2	3	2	3

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table X presents the analogue of Table I for the two discrete economies with $\Delta = 9$ and $\Delta = 1$. When $\Delta = 9$, we find no significant relationship between a firm's employment and local weighted employment. When $M = 3$, there is a negative but statistically insignificant relationship between a firm's employment and its local weighted population. In other words, more productive firms seem to have a larger footprint in less profitable locations. This last result showcases how, for high values of Δ, the predictions implied by Proposition 5 can fail to explain the behavior of firms. In contrast, when $\Delta = 1$ the estimated relationship is positive and significant for both values of M. Thus, for $\Delta = 1$, the results are consistent with the predictions of the proposition, and our empirical findings in Table I.

Of course, we acknowledge that this is just an example and that, in other cases, we might need to use even lower values of Δ to obtain allocations in the discrete economy that exhibit the properties derived for our limit economy. Unfortunately, the inability to reliably solve numerical examples with more locations restricts our ability to test the accuracy of our continuous limit in more complex examples. In any case, using our insights for environments where firms have very few plants, seems unwarranted.